# ZETA DETERMINANTS ON MANIFOLDS WITH BOUNDARY

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Abstract. We study the $\zeta$-determinant of global boundary problems of APS-type through a general theory for relative spectral invariants. In particular, we compute the $\zeta$-determinant for Dirac-Laplacian boundary problems in terms of a scattering Fredholm determinant over the boundary.

Contents



## 1. Introduction

The purpose of this paper is to study the $\zeta$-function regularized determinant for the Dirac Laplacian of an Atiyah-Patodi-Singer (APS)-type boundary problem. We do so within a general framework for studying relative global spectral invariants on manifolds with boundary. Despite the primacy of the determinant of the Dirac Laplacian over closed manifolds, relatively little has been known in the case of global boundary problems of APS-type. Geometric index theory of such boundary value problems began with the index formula [1]

$$\mathrm{ind}\,(D_{\Pi_{\geq}}) = \int_X \omega(D) - \frac{\eta(\mathcal{D}_Y) + \dim \mathrm{Ker}(\mathcal{D}_Y)}{2}\ . \tag{1.1}$$

Here $D : C^\infty(X, E^1) \longrightarrow C^\infty(X, E^2)$ is a first-order elliptic differential operator of Dirac-type acting over a compact manifold $X$ with boundary $\partial X = Y$. Near the boundary $D$ is assumed to act in the tangential direction via a first-order self-adjoint elliptic operator $\mathcal{D}_Y$ over $Y$. A boundary problem $D_B = D$ is defined by restricting the domain of $D$ to those sections whose boundary values lie in the kernel of a suitable order zero pseudodifferential operator $B$ on the space of boundary fields. APS-type boundary problems refer to the case where $B$ is, in a suitable sense, 'comparable' to the projection $\Pi_\geq$ onto the eigenspaces of $\mathcal{D}_Y$ with non-negative eigenvalue. The other ingredients in (1.1) are the index density $\omega(D)$ restricted from the closed double, and the eta-invariant $\eta(\mathcal{D}_Y)$, defined as the meromorphically continued value at $s = 0$ of $\eta(\mathcal{D}_Y, s) = \mathrm{Tr}\,(\mathcal{D}_Y|\mathcal{D}_Y|^{-s-1})$.

A striking consequence of (1.1) is that, in contrast to the case of closed manifolds, the index of APS-type boundary problems is not a homotopy invariant. If, however, we restrict the class of boundary conditions to a suitable classifying space for even K-theory, then the index does become a homotopy invariant of the boundary condition. An appropriate parameter space is a restricted Grassmannian $Gr(D)$ of pseudodifferential operator ($\psi$do) projections $P$ on the $L^2$ boundary fields which are comparable to $\Pi_\geq$. Such a Grassmannian has homotopy type $\mathbb{Z} \times BU$ and its connected components $Gr^{(r)}(D)$ are labeled by the index $r = \mathrm{ind}\,(D_P)$. One moves between the different components according to the *relative-index formula*

$$\mathrm{ind}\,(D_{P_1}) - \mathrm{ind}\,(D_{P_2}) = \mathrm{ind}\,(P_2, P_1)\ , \tag{1.2}$$

where $(P_2, P_1) := P_1 \circ P_2 : \mathrm{ran}\,(P_2) \to \mathrm{ran}\,(P_1)$ acts between the ranges of the projections $P_1, P_2 \in Gr(D)$.

The identity (1.2) depends on two decisive properties of global boundary problems over $Gr(D)$. The first is analytic: the restriction to the boundary of the infinite-dimensional solution space $\mathrm{Ker}(D)$, to the subspace $H(D) := \mathrm{Ker}(D)_{|Y}$ of boundary sections, is a continuous bijection, with canonical left inverse defined by the Poisson operator of $D$. The resulting isomorphism between the finite-dimensional kernel of $D_P$ and the kernel of the boundary operator $S(P) = P \circ P(D) : H(D) \to \mathrm{ran}\,(P)$, where $P(D)$ is the Calderon projection, and similarly, between the kernels of the adjoint operators, means that

$$\mathrm{ind}\,(D_P) = \mathrm{ind}\,(S(P))\ . \tag{1.3}$$

The second property is geometric: $Gr(D)$ is a homogeneous manifold, acted on transitively by an infinite-dimensional restricted general linear group on the space of boundary fields, resulting in the identity $\mathrm{ind}\,(P_1, P_2) + \mathrm{ind}\,(P_2, P_3) = \mathrm{ind}\,(P_1, P_3)$ for any $P_1, P_2, P_3 \in Gr(D)$. Then (1.2) follows trivially from (1.3).

Although the relative index formula is quite classical, these two properties resonate more powerfully when one turns to the harder problem of computing for APS-type boundary problems the spectral and differential geometric invariants familiar from closed manifolds. A precise understanding of these invariants is crucial for a

direct approach to a geometric index theory of global boundary problems of APS-type parallel to that for closed manifolds [2, 3]. An important but rather different perspective is provided by the $b$-calculus developed by Melrose [18].

One spectral invariant that certainly is well understood is the $\eta$-invariant $\eta(D_P)$ for self-adjoint global boundary problems over odd-dimensional manifolds. As a result of its semi-local character the $\eta$-invariant obeys a strikingly simple (to state) additivity property with respect to a partition of a closed manifold [5, 14, 20, 34]. In this case the homogeneous structure of the 'self-adjoint' Grassmannian takes a much simpler form: the range of any such $P$ occurs as the graph of a unitary isomorphism $T : F^+ \to F^-$, where $F^\pm$ denote the spaces of boundary chiral spinor fields [24].

The next invariant in the spectral hierarchy remains far more mysterious. The spectral $\zeta$-function of the Laplacian boundary problem $\Delta_P = (D_P)^* D_P$ is defined for $\mathrm{Re}(s) >> 0$ by the operator trace

$$\zeta(\Delta_P, s) = \mathrm{Tr}\,(\Delta_P^{-s}) \ ,$$

where we assume that $D_P$ is invertible. Recent results of Grubb, following earlier joint work with Seeley [10, 11, 12], show that for $P$ in the 'smooth' Grassmannian $Gr_\infty(D)$ (see §3), $\zeta(\Delta_P, s)$ has a meromorphic continuation to $\mathbb{C}$ which is regular at $s = 0$. This means there is a well-defined regularized $\zeta$-determinant of the Laplacian

$$\det{}_\zeta(\Delta_P) = \exp\left(-\frac{d}{ds}_{|s=0}\zeta(\Delta_P, s)\right) \ . \tag{1.4}$$

Owing to its highly non-local nature $\det_\zeta(\Delta_P)$ is a hopelessly difficult invariant to compute. There is, on the other hand, a quite different but also completely canonical regularization of the determinant of $\Delta_P$ as the Fredholm determinant of the boundary 'Laplacian'

$$S(P)^* S(P) : H(D) \longrightarrow H(D) \ .$$

The Fredholm determinant $\det_F$ is defined for operators on a Hilbert space differing from the identity by an operator of trace class and is the natural extension to infinite-dimensional spaces of the usual determinant in finite dimensions. Its analytical status, however, is essentially opposite to that of (1.4). More precisely, the $\zeta$-determinant is *not* an extension of the Fredholm determinant—operators with Fredholm determinants do not have $\zeta$-determinants, operators with $\zeta$-determinants do not have Fredholm determinants[1]. A more subtle fact is nevertheless true: the *relative* $\zeta$-determinant is a true extension of the Fredholm determinant. Here a relative regularized determinant means a regularization of the ratio $\det A_1/\det A_2$ for 'comparable' operators $A_1, A_2$ (see §2). Thus, any pair of determinant class (= Id + Trace Class) operators have a well-defined relative $\zeta$-determinant. Moreover, there is a, roughly converse, 'relativity principle for determinants' which states that

[1] We consider here infinite-dimensional Hilbert spaces.

ratios of $\zeta$-determinants for certain preferred classes of unbounded operators can be written canonically in terms of Fredholm determinants.

Applied to global boundary problems the relativity principle for determinants is a restatement of the fact that in order to define a topologically meaningful Grassmannian one must do so relative to a basepoint projection. (Dimension one is an exception since we are reduced in this case to the usual finite-dimensional Grassmannian, no basepoint is needed and for this reason explicit formulas for the $\zeta$-determinant of ordinary differential operators are possible. See §5.) This is familiar in physics in the quantization of Fermions where the basepoint corresponds to the Dirac sea splitting into positive and negative energy modes (the APS splitting). The application to the determinant is a well known, if imprecise, idea in physics folklore used extensively in defining path integrals in QFT and String Theory.

In §2 we prove a precise formulation of the relativity principle for determinants adequate for our purposes here. The main result in this paper is Theorem A in the following table. The table summarizes relative formulas for the key spectral invariants.

| Invariant | Relative Formula |
|---|---|
| Index | $\mathrm{ind}\,(D_{P_1}) - \mathrm{ind}\,(D_{P_2}) = \mathrm{ind}\,(P_2, P_1)$ |
| Eta-invariant: Odd-dimensions | $\eta(D_{P_1}, 0) - \eta(D_{P_2}, 0) = \frac{1}{\pi i} \log \det_F(T_2^{-1} T_1)$ |
| Zeta-determinant: Odd-dimensions | $\frac{\det_\zeta D_{P_1}}{\det_\zeta D_{P_2}} = \frac{\det_F\left(\frac{1}{2}(I+T_1^{-1}K)\right)}{\det_F\left(\frac{1}{2}(I+T_2^{-1}K)\right)}$ |
| Laplacian Zeta-determinant | **Theorem A**: $\frac{\det_\zeta(\Delta_{P_1})}{\det_\zeta(\Delta_{P_2})} = \frac{\det_F(S(P_1)^* S(P_1))}{\det_F(S(P_2)^* S(P_2))}$ |

The third formula in the table is Theorem (0.1) of [28] for self-adjoint global boundary problems $D_{P_1}, D_{P_2}$ over a compact odd-dimensional manifold. The operators $T_i, K : F^+ \to F^-$ are the boundary unitary isomorphisms discussed earlier, with $H(D) = \mathrm{graph}(K)$. Because, in this case, the $\eta$-invariant is essentially the phase of the determinant, the second formula, which holds $\mod 2\mathbf{Z}$, is an easy corollary when the operators are invertible [14]. Theorem A holds for general Dirac-type operators and in all dimensions. Notice, furthermore, that it is stated invariantly, independently of the choice of 'coordinates' $T_i$—in §3.4 we explain how the second and third formulas in the table are derived from the invariant general formulas proved in §2 for the relative $\eta$-invariant and $\zeta$-determinant.

The relative determinant formulas in the table encode a certain spectral duality between the rapidly diverging eigenvalues of the global boundary problems and the eigenvalues of the boundary Laplacians, which converge rapidly to 1. The point

being that, for comparable global boundary problems, taking quotients produces arithmetically similar behaviour.

This extends to a differential geometric duality between smooth families of global boundary problems $(\mathbb{D},\mathbb{P}) = \{D^b_{P_b} \mid b \in B\}$ parameterized by a manifold $B$ and the corresponding family of boundary operators $\mathbb{S}(\mathbb{P}) = \{S(P_b) = P_b \circ P(D^b) \mid b \in B\}$. Each family has an associated index bundle and determinant line bundle. Geometrically the regularized determinants $\det_\zeta \Delta_P$ and $\det_F(S(P)^*S(P))$ define the so-called Quillen metric [2, 3] and canonical metric [25] on the respective determinant line bundles. For example, if $D$ is held fixed and $P$ allowed to vary in the parameter manifold $B = Gr_\infty(D)$ the canonical metric is just the 'Fubini-Study' metric of [22] over the restricted Grassmannian. From this view point, Theorem A expresses the relative equality of these metrics with respect to the obvious determinant line bundle isomorphism $\mathrm{DET}\,(\mathbb{D},\mathbb{P}) \cong \mathrm{DET}\,(\mathbb{S}(\mathbb{P}))$. The following table lists relative geometric index theory formulas for families of APS-type boundary problems:

| Invariant | Relative Formula |
|---|---|
| Index bundle | $\mathrm{Ind}\,(\mathbb{D},\mathbb{P}_1) - \mathrm{Ind}\,(\mathbb{D},\mathbb{P}_2) = \mathrm{Ind}\,(\mathbb{P}_2,\mathbb{P}_1)$ |
| Determinant line bundle | $\mathrm{DET}\,(\mathbb{D},\mathbb{P}_1) \otimes \mathrm{DET}\,(\mathbb{D},\mathbb{P}_2)^* \cong \mathrm{DET}\,(\mathbb{P}_2,\mathbb{P}_1)$ |
| Zeta curvature | $\Omega^\zeta_1 - \Omega^\zeta_2 = \Omega^{\mathcal{C}}_1 - \Omega^{\mathcal{C}}_2$ |

The relative index bundle formula is taking place in $K^0(B)$. For a functional analytic proof see [6]. The determinant line bundle isomorphism is explained in [25], where, essentially, the definition is given of the canonical curvature form $\Omega^{\mathcal{C}}$ on $\mathrm{DET}\,(\mathbb{S}(\mathbb{P}))$. For the construction of the $\zeta$-connection on $\mathrm{DET}\,(\mathbb{D},\mathbb{P})$ with curvature form $\Omega^\zeta$ and proofs of all three identities, see [27].

Notice that by setting $P_2 = P(D)$, the relative formulas in the tables may be re-expressed as an interior term and a boundary correction term.

There is an essentially immediate application of the methods here to non-compact manifolds. For a closely related detailed study of the Laplacian we refer to the seminal paper of Muller [21]. For an account of how determinants of global boundary problems fit into the framework of TQFT we refer to [19]. The results of this paper were announced in [26].

The paper is organized as follows.

In §2 we prove a precise form of the relativity principle for determinants using regularized limits of Fredholm scattering determinants (Theorem 2.5). In §2.1 we explain how this is related to the heat operator regularization of the determinant—the

more usual scenario for studying scattering determinants. The relative eta invariant for comparable self-adjoint operators requires a somewhat different treatment. In §2.2 we prove a general formula for the relative eta invariant as the difference of two scattering determinant limits. §2.3 is concerned with a general multiplicativity property for the zeta determinant.

In §3 we first review some analytic facts about first-order global boundary problems which will be needed as we proceed. We explain how the scattering determinant arises canonically in terms of natural isomorphisms between the various determinant lines. In Theorem 3.13 we prove an explicit formula for the relative zeta determinant of first-order global boundary problems. As an example, we use this formula to give a new derivation of Theorem (0.1) of [28].

In §4 we use Theorem 2.5 to prove a formula for the relative zeta determinant of the Dirac Laplacian in terms of an equivalent first-order system. Methods from §3 then reduce this to the equality in Theorem A.

In §5 we give a new proof using our methods of the results in [15] for differential operators in dimension one. In this sense, the results of this paper may be regarded as the extension of [15] to all dimensions.

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## 2. Regularized Limits and the Relative Zeta Determinant

Let $A_1, A_2$ be invertible closed operators on a Hilbert space $H$ with a common spectral cut $R_\theta = \{re^{i\theta} \mid r \geq 0\},\ \theta \in [0, 2\pi)$. This supposes $\delta, \varrho > 0$ such that the resolvents $(A_i - \lambda)^{-1}$ are holomorphic in the sector

$$\Lambda_\theta = \{z \in \mathbb{C} \mid |\arg(z) - \theta| < \delta \ \text{ or } \ |z| < \varrho\} \tag{2.1}$$

and such that the operator norms $\|(A_i - \lambda)^{-1}\|$ are $O(|\lambda|^{-1})$ as $\lambda \to \infty$ in $\Lambda_\theta$. For $\mathrm{Re}(s) > 0$ one then has the complex powers first studied by Seeley [30]

$$A_i^{-s} = \frac{i}{2\pi} \int_C \lambda^{-s} (A_i - \lambda)^{-1} \, d\lambda \ , \tag{2.2}$$

where

$$\lambda^{-s} = |\lambda|^{-s} e^{-is\arg(\lambda)}, \qquad \theta - 2\pi \leq \arg(\lambda) < \theta \ , \tag{2.3}$$

is the branch of $\lambda^{-s}$ defined by the spectral cut $R_\theta$, and $C = C_\theta$ is the negatively oriented contour $C_{\theta,\downarrow} \cup C_{\rho,\theta,\theta-2\pi} \cup C_{\theta-2\pi,\uparrow}$, with

$$C_{\phi,\downarrow} = \{\lambda = re^{i\phi} \mid \infty > r \geq \rho\} \ , \quad C_{\rho,\phi,\phi'} = \{\rho e^{i\theta'} \mid \phi \geq \theta' \geq \phi'\} \ , \tag{2.4}$$

$$C_{\phi,\uparrow} = \{\lambda = re^{i\phi} \mid \rho \leq r < \infty\} \ ,$$

and $\rho < \varrho$. We assume there is a real $\alpha_0$ such that the operators

$$\partial_\lambda^m (A_i - \lambda)^{-1} = m!(A_i - \lambda)^{-m-1}$$

are trace class for $m > -\alpha_0$, with asymptotic expansions as $\lambda \to \infty$ in $\Lambda_\theta$

$$\mathrm{Tr}\,(\partial_\lambda^m (A_i - \lambda)^{-1}) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{1} a_{j,k}^{(i)}(m).(-\lambda)^{-\alpha_j - m}\log^k(-\lambda)\ , \tag{2.5}$$

where $0 < m+\alpha_0 < \ldots < m+\alpha_j \nearrow +\infty$. Here, $\log = \log_\theta$ is the branch of the logarithm specified by (2.3); changing $\theta$ may change the coefficients $a_{j,k}^{(i)}(m)$. Since $\lambda^{k-s}\partial_\lambda^{k-1}(A_i-\lambda)^{-1} \to 0$ as $\lambda \to \infty$ along $C$ for $\mathrm{Re}(s) > 0$, we can integrate by parts in (2.2) to obtain

$$A_i^{-s} = \frac{1}{(s-1)\ldots(s-m)}.\frac{i}{2\pi}\int_C \lambda^{m-s}\,\partial_\lambda^m (A_i-\lambda)^{-1}\,d\lambda\ . \tag{2.6}$$

From (2.5) and (2.6), the operators $A_i^{-s}$ are trace class in the half-plane $\mathrm{Re}(s) > 1+m$ and we can define there the spectral zeta functions of $A_1$, $A_2$

$$\zeta_\theta(A_i, s) = \mathrm{Tr}\,A_i^{-s}, \qquad\qquad \mathrm{Re}(s) > 1+m\ .$$

Substituting the asymptotic expansion (2.5) in

$$\zeta_\theta(A_i, s) = \frac{1}{(s-1)\ldots(s-m)}.\frac{i}{2\pi}\int_C \lambda^{m-s}\,\mathrm{Tr}\,(\partial_\lambda^m (A_i-\lambda)^{-1})\,d\lambda\ . \tag{2.7}$$

yields the meromorphic continuation of $\zeta_\theta(A_i, s)$ to all of $\mathbb{C}$ with singularity structure

$$\frac{\pi}{\sin(\pi s)}\zeta_\theta(A_i, s) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{1}\frac{\tilde{a}_{j,k}^{(i)}}{(s+\alpha_j-1)^{k+1}}\ , \tag{2.8}$$

where, independently of m,

$$\tilde{a}_{j,k}^{(i)} \approx \mu_k\Gamma(\alpha_j)\Gamma(\alpha_j+m)^{-1}a_{j,k}^{(i)}(m)\ , \tag{2.9}$$

$\mu_k = (1+i(\theta-\pi))^k$ and $\Gamma(s)$ is the Gamma function (see [29], [12] Prop 2.9, here generalized to arbitrary $\theta$).

*Notation*: In equation (2.9) $\approx$ indicates that

$$\frac{\Gamma(\alpha_j)\Gamma(\alpha_j+m)^{-1}a_{j,0}^{(i)}(m)}{s+\alpha_j-1} + \frac{\mu_1\Gamma(\alpha_j)\Gamma(\alpha_j+m)^{-1}a_{j,1}^{(i)}(m)}{(s+\alpha_j-1)^2}$$

gives the full pole structure at $s = 1-\alpha_j$. A function defined in the sector $\Lambda_\theta$ has an asymptotic expansion

$$f(\lambda) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{1} c_{j,k}(-\lambda)^{-\beta_j}\log^k(-\lambda) + c(-\lambda)^{-\nu}$$

as $\lambda \to \infty$ with $\beta_j \nearrow +\infty, \nu > 0$ means that for any $\epsilon > 0$ and $N$ with $\beta_N > \nu$,

$$f(\lambda) = \sum_{j=0}^{N-1}\sum_{k=0}^{1} c_{j,k}(-\lambda)^{-\beta_j}\log^k(-\lambda) + c(-\lambda)^{-\nu} + O(|\lambda|^{-\beta_N+\epsilon})\ ,$$

for $\lambda$ sufficiently large, while a function $g$ on $\mathbb{C}$ has singularity structure

$$g(s) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{1} \frac{d_{j,k}}{(s+\gamma_j-1)^k}$$

means that

$$g(s) = \sum_{j=0}^{N-1}\sum_{k=0}^{1} \frac{d_{j,k}}{(s+\gamma_j-1)^k} + h_N(s) \ ,$$

with $h_N$ holomorphic for $1-\gamma_N < \mathrm{Re}(s) < N+1$.

At any rate, (2.8) implies that the term with coefficient $a_{j,k}^{(i)}(m)$ in the resolvent trace expansion (2.5) corresponds to a pole of $\frac{\pi}{\sin(\pi s)}\zeta_\theta(A_i,s)$ at $s = 1-\alpha_j$ of order $k+1$. In particular, since $\frac{\sin(\pi s)}{\pi} = s + O(s^3)$ around $s=0$, if

$$\tilde{a}_{J,1}^{(1)} = 0 = \tilde{a}_{J,1}^{(2)} \ , \qquad \alpha_J = 1 \ , \tag{2.10}$$

then (2.8) implies the $\zeta_\theta(A_i,s)$ have no pole at $s=0$ and

$$\zeta_\theta(A_i,0) = \tilde{a}_{J,0}^{(i)} = \frac{a_{J,0}^{(i)}(m)}{m!} \ . \tag{2.11}$$

The regularity at $s=0$ means we can define the $\zeta$-determinants

$$\det{}_{\zeta,\theta}A_1 = e^{-\zeta_\theta'(A_1,0)} \ , \qquad \det{}_{\zeta,\theta}A_2 = e^{-\zeta_\theta'(A_2,0)} \ ,$$

where $\zeta_\theta' = d/ds(\zeta_\theta)$. If (2.10) holds we refer to each of $A_1, A_2$ as *$\zeta$-admissible.* Thus, for example, elliptic $\psi$dos of order $d>0$ over a closed manifold of dimension $n$ are $\zeta$-admissible with $a_{j,k}^{(i)}(m)$ locally determined, $\alpha_j = (j-n)/d$, so $J = n+d$; for *differential* operators $a_{j,1}^{(i)}(m) = 0$ (no log terms). In the following we do not assume that the operators $A_i$ are $\zeta$-admissible unless explicitly stated.

**Definition 2.1.** *We refer to a pair $(A_1,A_2)$ of invertible closed operators on $H$ with spectral cut $R_\theta$ as $\zeta$-comparable if for $\lambda\in\Lambda_\theta$ :*
**(I)** *The relative resolvent $(A_1-\lambda)^{-1} - (A_2-\lambda)^{-1}$ is trace class and*

$$\mathrm{Tr}\left((A_1-\lambda)^{-1} - (A_2-\lambda)^{-1}\right) = -\frac{\partial}{\partial\lambda}\log\det{}_F\mathcal{S}_\lambda \ . \tag{2.12}$$

*Here the 'scattering' operator $\mathcal{S}_\lambda = \mathcal{S}_\lambda(A_1,A_2)$ is an operator of the form $Id + W_\lambda$ on a Hilbert space $\mathcal{H}_\lambda \subseteq H$ with $W_\lambda$ of trace class, so that $\mathcal{S}_\lambda$ has a Fredholm determinant $\det_F\mathcal{S}_\lambda := 1 + \sum_{k\geq 1}\mathrm{Tr}\,(\wedge^k W_\lambda)$ taken on $\mathcal{H}_\lambda$.*
**(II)** *There is an asymptotic expansion as $\lambda\to\infty$*

$$\mathrm{Tr}\left((A_1-\lambda)^{-1} - (A_2-\lambda)^{-1}\right) \sim \sum_{\substack{j=0\\ j\neq J}}^{\infty}\sum_{k=0}^{1} b_{j,k}(-\lambda)^{-\alpha_j}\log^k(-\lambda) + b_{J,0}(-\lambda)^{-1} \ , \tag{2.13}$$

*where $0<\alpha_0<\ldots<\alpha_j \nearrow +\infty$ and $\alpha_J = 1$.*

**Remark 2.2.** *(1) If $\mathcal{H} := \cup_\lambda \mathcal{H}_\lambda$ forms a (trivializable) vector bundle, then the right-side of* (2.12) *can be written* $-\mathrm{Tr}\,(\mathcal{S}_\lambda^{-1}\, d_\lambda \mathcal{S}_\lambda)$*, where $d_\lambda$ is a covariant derivative on* $\mathrm{Hom}(\mathcal{H}, H)$*. There is a canonical choice for $d_\lambda$ induced from the covariant derivative $\nabla_{\partial/\partial\lambda} = P(\lambda)\cdot(\partial/\partial\lambda)\cdot P(\lambda)$ on $\mathcal{H}$, with $P(\lambda)$ the projection on $H$ with range $\mathcal{H}_\lambda$.*

*(2) If an expansion* (2.13) *exists, then $\alpha_j > 0$ since $A_1^{-1} - A_2^{-1}$ is trace class.*

If $A_1, A_2$ are $\zeta$-comparable, then $A_1^{-s} - A_2^{-s}$ is trace class for $\mathrm{Re}(s) > 1$. Hence we define the *relative spectral $\zeta$-function* by

$$\zeta_\theta(A_1, A_2, s) = \mathrm{Tr}\,(A_1^{-s} - A_2^{-s})\ , \qquad \mathrm{Re}(s) > 1\ . \tag{2.14}$$

In view of (2.12) we have

$$\begin{aligned} \zeta_\theta(A_1, A_2, s) &= \frac{i}{2\pi}\int_C \lambda^{-s}\,\mathrm{Tr}\,((A_1-\lambda)^{-1} - (A_2-\lambda)^{-1})\, d\lambda \\ &= -\frac{i}{2\pi}\int_C \lambda^{-s}\,\frac{\partial}{\partial\lambda}\log\det{}_F \mathcal{S}_\lambda\, d\lambda\ . \end{aligned} \tag{2.15}$$

$\zeta_\theta(A_1, A_2, s)$ thus extends holomorphically to $\mathrm{Re}(s) > 1 - \alpha_0$ while the asymptotic expansion (2.13) defines the meromorphic continuation to $\mathbb{C}$ with singularity structure

$$\Gamma(s)\zeta_\theta(A_1, A_2, s) \sim \sum_{\substack{j=0\\ j\neq J}}^{\infty}\sum_{k=0}^{1}\frac{\tilde{b}_{j,k}}{(s+\alpha_j-1)^k} + \frac{\tilde{b}_{J,0}}{s}\ , \tag{2.16}$$

where

$$\tilde{b}_{j,k} = \mu_k\Gamma(\alpha_j)^{-1} b_{j,k}, \qquad \tilde{b}_{J,0} = b_{J,0} = \zeta_\theta(A_1, A_2, 0)\ . \tag{2.17}$$

Since $b_{J,1} = 0$ in (2.13) then $\zeta_\theta(A_1, A_2, s)$ is regular at $s = 0$ and we can define the *relative $\zeta$-determinant* by

$$\det{}_{\zeta,\theta}(A_1, A_2) = e^{-\zeta_\theta'(A_1, A_2, 0)}\ . \tag{2.18}$$

No assumption is made on the existence or regularity of $\zeta_\theta(A_i, s)$. If $A_1, A_2$ are $\zeta$-admissible and (I) of Definition (2.1) holds, then (2.5), (2.10) imply as $\lambda \to \infty$ in $\Lambda_\theta$

$$\begin{aligned} &\partial_\lambda^m \mathrm{Tr}\,((A_1-\lambda)^{-1} - (A_2-\lambda)^{-1}) \sim \\ &\sum_{\substack{p=0\\ p\neq J}}^{\infty}\sum_{k=0}^{1}(a_{p,k}^{(1)}(m) - a_{p,k}^{(2)}(m))(-\lambda)^{-\alpha_p - m}\log^k(-\lambda) + (a_{J,0}^{(1)}(m) - a_{J,0}^{(2)}(m))(-\lambda)^{-1-m}\ , \end{aligned} \tag{2.19}$$

with, by $\zeta$-comparability, $\alpha_p > 0$—for, the relative resolvent trace is then $O(|\lambda|^{-\varepsilon})$ as $\lambda \to \infty$, some $\varepsilon > 0$, and hence $a_{j,k}^{(1)}(m) - a_{j,k}^{(2)}(m) = 0$ in (2.5) for $\alpha_j \leq 0$, while in (2.19) $p = j - \max\{j \mid \alpha_j \leq 0\} + 1$, resulting in the regularity of $\zeta_\theta(A_1, A_2, s)$ for $\mathrm{Re}(s) > 1 - \varepsilon$. With $\alpha_p > 0$ we can integrate (2.19) to obtain for $\lambda \to \infty$ an asymptotic expansion of the form (2.13).

The $b_{j,k}$ are related to the coefficients in (2.19) via universal constants; in (2.16)

$$\tilde{b}_{j,k} \approx \tilde{a}^{(1)}_{j,k} - \tilde{a}^{(2)}_{j,k} \ , \tag{2.20}$$

and in particular

$$\zeta_\theta(A_1, A_2, 0) = \tilde{b}_{J,0} = \tilde{a}^{(1)}_{J,0} - \tilde{a}^{(2)}_{J,0} = \zeta_\theta(A_1, 0) - \zeta_\theta(A_2, 0) \ . \tag{2.21}$$

More generally:

**Lemma 2.3.** *If $A_1, A_2$ are $\zeta$-admissible operators such that (I) of Definition (2.1) holds, then $A_1, A_2$ are $\zeta$-comparable and as meromorphic functions on $\mathbb{C}$*

$$\zeta_\theta(A_1, A_2, s) = \zeta_\theta(A_1, s) - \zeta_\theta(A_2, s) \ . \tag{2.22}$$

*Proof.* The first statement is proved above. For $\mathrm{Re}(s) > 1 - \alpha_0$, (2.22) is obvious. Elsewhere, from (2.17), or (2.19), (2.13), the left and right sides of (2.22) have the same singularity structure, hence $\zeta_\theta(A_1, A_2, s) - \zeta_\theta(A_1, s) + \zeta_\theta(A_2, s)$ is a holomorphic continuation of zero from $\mathrm{Re}(s) > 1 - \alpha_0$ to all of $\mathbb{C}$, and is therefore identically zero. □

To compute $\det_{\zeta,\theta}(A_1, A_2)$ in terms of the scattering matrix we need to know more about the asymptotic behaviour of $\mathcal{S}_\lambda$.

**Lemma 2.4.** *Let $f$ be a differentiable function in the sector $\Lambda_\theta$ with an asymptotic expansion as $\lambda \to \infty$*

$$-\frac{\partial f}{\partial \lambda} \sim \sum_{j=0}^{\infty} \sum_{k=0}^{1} c_{j,k} (-\lambda)^{-\beta_j} \log^k(-\lambda) + c_0 (-\lambda)^{-1} \tag{2.23}$$

*with $\beta_j \nearrow +\infty$ and $\beta_j \neq 1$. Then*

$$c_f(\lambda) := f(\lambda) - \sum_{j=0}^{r} \sum_{k=0}^{1} \frac{c_{j,k}}{1-\beta_j} \left( (-\lambda)^{-\beta_j+1} \log^k(-\lambda) - \frac{k}{1-\beta_j} (-\lambda)^{-\beta_j+1} \right) - c_0 \log(-\lambda), \tag{2.24}$$

*where $r = \max\{k \mid \beta_k < 1\}$, converges uniformly as $\lambda \to \infty$. Denoting this limit by*

$$c_1 := \lim^{\theta}_{\lambda\to\infty} c_f(\lambda) \ ,$$

*(the $\theta$ indicating the limit is taken in the sector $\Lambda_\theta$), there is an asymptotic expansion as $\lambda \to \infty$ in $\Lambda_\theta$*

$$f(\lambda) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{1} \frac{c_{j,k}}{1-\beta_j} \left( (-\lambda)^{-\beta_j+1} \log^k(-\lambda) - \frac{k}{1-\beta_j} (-\lambda)^{-\beta_j+1} \right) + c_0 \log(-\lambda) + c_1 \ . \tag{2.25}$$

*Proof.* Let $\lambda_0 \in R_\varphi$ with $\varphi = \arg(\lambda)$ and $|\lambda| < |\lambda_0|$. Choosing $|\lambda|$ sufficiently large so that (2.23) holds, then

$$|c_f(\lambda) - c_f(\lambda_0)| \leq \int_{|\lambda_0|}^{|\lambda|} \left| \frac{\partial c_f}{\partial \mu} \right| d\mu = \int_{|\lambda_0|}^{|\lambda|} \Big| \frac{\partial f}{\partial \mu} + \sum_{j=0}^{r} \sum_{k=0}^{1} c_{j,k} (-\mu)^{-\beta_j} \log^k(-\mu) + c_0 (-\mu)^{-1} \Big| \, d\mu$$

$$= \int_{|\lambda_0|}^{|\lambda|} O(\mu^{-\beta_{r+1}})\, d\mu \leq C|\lambda|^{-\beta_{r+1}+1} \ .$$

Since $\beta_{r+1} > 1$, then $(c_f(\lambda))$ is convergent by the Cauchy criterion.

Integrating (2.23) between $\lambda$ and $\lambda_0$, we obtain for large $|\lambda|, |\lambda_0|$ and any $\epsilon > 0$

$$f(\lambda) = \sum_{j=0}^{N-1} \sum_{k=0}^{1} \frac{c_{j,k}}{1-\beta_j} \left( (-\lambda)^{-\beta_j+1} \log^k(-\lambda) - \frac{k}{1-\beta_j} (-\lambda)^{-\beta_j+1} \right)$$

$$+ c_0 \log(-\lambda) + O(|\lambda|^{-\beta_N+1+\epsilon}) + c_f(\lambda_0) + O(|\lambda_0|^{-\beta_N+1+\epsilon}) \ .$$

Letting $\lambda_0 \to \infty$ we reach the conclusion. □

Applying Lemma 2.4 to $\varphi(\lambda) = \log \det_F \mathcal{S}_\lambda$ and from (2.12), (2.13), we see that as $\lambda \to \infty$ in $\Lambda_\theta$ there is an asymptotic expansion

$$\log \det_F \mathcal{S}_\lambda \sim \sum_{\substack{j=0 \\ j\neq J}}^{\infty} \sum_{k=0}^{1} b'_{j,k} (-\lambda)^{-\alpha_j+1} \log^k(-\lambda) + b_{J,0} \log(-\lambda) + c_{\text{rel}} \ , \tag{2.26}$$

where $b'_{j,0} = b_{j,0}(1-\alpha_j)^{-1} - b_{j,1}(1-\alpha_j)^{-2}$, $b'_{j,1} = b_{j,1}(1-\alpha_j)^{-1}$ $(j \neq J)$, and the constant term is

$$c_{\text{rel}} = \lim^\theta_{\lambda\to\infty} [\log \det_F \mathcal{S}_\lambda - b_{J,0} \log(-\lambda) - \sum_{j=0}^{J-1} \sum_{k=0}^{1} b'_{j,k} (-\lambda)^{-\alpha_j+1} \log^k(-\lambda)] \ . \tag{2.27}$$

The regularized limit of a function in the sector $\Lambda_\theta$ with an asymptotic expansion $f(\lambda) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{1} \mathsf{c}_{jk} (-\lambda)^{-\beta_j} \log^k(-\lambda) + \mathsf{c}_0 \log(-\lambda) + \mathsf{c}_1$ as $\lambda \to \infty$, where $\beta_j \nearrow +\infty$ and $\beta_j \neq 0$, picks out the constant term in the expansion

$$\text{LIM}^\theta_{\lambda\to\infty} f(\lambda) = \mathsf{c}_1 \ .$$

We have (with $\mathcal{S} := \mathcal{S}_0$):

**Theorem 2.5.** *For $\zeta$-comparable operators $A_1, A_2$*

$$\det_{\zeta,\theta}(A_1, A_2) = \det_F \mathcal{S} \,.\, e^{-\text{LIM}^\theta_{\lambda\to\infty} \log \det_F \mathcal{S}_\lambda} \ . \tag{2.28}$$

*With $\zeta_{\text{rel}}(0) := \zeta_\theta(A_1, A_2, 0)$, one has*

$$\text{LIM}^\theta{}_{\lambda\to\infty} \log \det_F \mathcal{S}_\lambda = \lim^\theta_{\lambda\to\infty} [\log \det_F \mathcal{S}_\lambda - \zeta_{\text{rel}}(0) \log(-\lambda) \tag{2.29}$$

$$- \sum_{j=0}^{J-1} \sum_{k=0}^{1} b'_{j,k} (-\lambda)^{-\alpha_j+1} \log^k(-\lambda)] \ .$$

*If $A_1, A_2$ are $\zeta$-admissible*

$$\det_{\zeta,\theta}(A_1, A_2) = \frac{\det_{\zeta,\theta} A_1}{\det_{\zeta,\theta} A_2} \ . \tag{2.30}$$

*Proof.* The identity (2.30) is immediate from Lemma 2.3, while (2.29) follows from (2.26), (2.27), and (2.17),(2.21).

To prove (2.28), since $\lambda^{-s}\log\det_F\mathcal{S}_\lambda \to 0$ at the ends of $C$ for $\mathrm{Re}(s) > 1-\alpha_0$, we can integrate by parts in (2.15) to obtain

$$\zeta_\theta(A_1, A_2, s) = sg(s) \tag{2.31}$$

and hence that

$$\zeta_\theta'(A_1, A_2, 0) = \frac{d}{ds}_{|s=0}(sg(s)|^{\mathrm{mer}}) \ , \tag{2.32}$$

where, with $f(\lambda) = \frac{\log\det_F\mathcal{S}_\lambda}{(-\lambda)}$,

$$g(s) = \frac{i}{2\pi}\int_C \lambda^{-s} f(\lambda)d\lambda \tag{2.33}$$

has a simple pole at $s = 0$ with residue $b_{J,0}$. The notation $h(s)|^{\mathrm{mer}}$ indicates the meromorphically continued function.

We carry out the meromorphic continuation of $\zeta_\theta(A_1, A_2, s)$ along the lines of [12] Prop 2.9 . First, $\log\det_F\mathcal{S}_\lambda$ is regular near $\lambda = 0$ and so $f(\lambda)$ is meromorphic there with Laurent expansion

$$f(\lambda) = \frac{\log\det_F\mathcal{S}}{(-\lambda)} + \sum_{j=0}^{\infty} b_j(-\lambda)^j \ . \tag{2.34}$$

Since $\int_C \lambda^{-1-s}d\lambda = 0$ for $\mathrm{Re}(s) > 0$, then

$$g(s) = \frac{i}{2\pi}\int_C \lambda^{-s} f_0(\lambda)d\lambda \ , \qquad f_0(\lambda) := f(\lambda) - \frac{\log\det_F\mathcal{S}}{(-\lambda)} \ . \tag{2.35}$$

For $1-\alpha_0 < \mathrm{Re}(s) < 1$, the circular part $C_{\rho,\theta,\theta-2\pi}$ of the contour $C$ can now be shrunk to the origin, which reduces $g(s)$ to

$$g(s) = \frac{\sin(\pi s)}{\pi}e^{i(\pi-\theta)(s-1)}\int_0^\infty r^{-s} f_0(re^{i\theta})\ dr \ , \tag{2.36}$$

using $e^{-is(\theta-2\pi)} - e^{-is\theta} = 2i\sin(\pi s).e^{is(\pi-\theta)}$. On the other hand, from (2.26) there is an asymptotic expansion as $\lambda \to \infty$ along $R_\theta$

$$f_0(\lambda) \sim c_0\frac{\log(-\lambda)}{(-\lambda)} + \frac{c_1 - \log\det_F\mathcal{S}}{(-\lambda)} + \sum_{\substack{j=0\\ j\neq J}}^{\infty}\sum_{k=0}^{1} b_{j,k}'(-\lambda)^{-\alpha_j}\log^k(-\lambda) \tag{2.37}$$

where $c_0 = b_{J,0} = \zeta_{rel}(0)$, $c_1 = c_{\mathrm{rel}}$. Hence, since $-\lambda = -re^{i\theta} = re^{i(\theta-\pi)}$ with respect to $R_\theta$, for any $\epsilon > 0, N > J+1$ we have as $r \to \infty$

$$\begin{aligned} f_0(re^{i\theta}) \ &= \ c_0\frac{\log(re^{i(\theta-\pi)})}{re^{i(\theta-\pi)}} + \frac{c_1 - \log\det_F\mathcal{S}}{re^{i(\theta-\pi)}} \\ &+ \ \sum_{\substack{j=0\\ j\neq J}}^{N-1}\sum_{k=0}^{1} b_{j,k}'r^{-\alpha_j}e^{i(\theta-\pi)\alpha_j}\log^k(re^{i(\theta-\pi)}) + O(r^{-\alpha_N+\epsilon}). \end{aligned} \tag{2.38}$$

Therefore

$$e^{-i(\theta-\pi)}e^{-is(\pi-\theta)}\frac{\pi}{\sin(\pi s)}g(s) = \int_0^1 \left[\sum_{j=0}^{N-1} b_j e^{i(\theta-\pi)j} r^{j-s} + r^{-s}O(r^N)\right]\ dr$$

$$+\int_1^\infty \Big[e^{-i(\theta-\pi)}c_0 r^{-s-1}\log(r) + e^{-i(\theta-\pi)}(c_1 + i(\theta-\pi)c_0 - \log\det{}_F\mathcal{S})r^{-s-1}$$

$$+\sum_{\substack{j=0\\ j\neq J}}^{N-1}\sum_{k=0}^{1} c_{j,k,\theta} r^{-\alpha_j - s}\log^k(r) + r^{-s}O(r^{-\alpha_N+\epsilon})\Big]\ dr$$

$$= -\sum_{j=0}^{N-1}\frac{b_j e^{i(\theta-\pi)j}}{s-j-1} + +\frac{e^{-i(\theta-\pi)}c_0}{s^2} + \frac{e^{-i(\theta-\pi)}(c_1 + i(\theta-\pi)c_0 - \log\det{}_F\mathcal{S})}{s}$$

$$+\sum_{\substack{j=0\\ j\neq J}}^{N-1}\sum_{k=0}^{1}\frac{c_{j,k,\theta}}{(s+\alpha_j-1)^k} + h_N(s)\ ,$$

where $h_N$ is holomorphic for $1-\alpha_N+\epsilon < \mathrm{Re}(s) < N+1$. Here we use the meromorphic extension to $\mathbb{C}$ of

$$\int_0^1 r^{j-s}\ dr = \frac{-1}{s-j-1}\ ,\qquad \mathrm{Re}(s) < j+1\ , \tag{2.39}$$

$$\int_1^\infty r^{\beta-s}\log^k(r)\ dr = \frac{1}{(s-\beta-1)^{k+1}}\ ,\qquad k=0,1\ ,\quad \mathrm{Re}(s) > \beta+1\ . \tag{2.40}$$

This implies the singularity structure

$$e^{-i(\pi-\theta)s}\frac{\pi}{\sin(\pi s)}g(s) \sim \frac{c_0}{s^2} + \frac{c_1 + i(\theta-\pi)c_0 - \log\det{}_F\mathcal{S}}{s} - \sum_{j=0}^{\infty}\frac{b_j}{s-j-1}$$

$$+\sum_{\substack{j=0\\ j\neq J}}^{\infty}\sum_{k=0}^{1}\frac{e^{i(\theta-\pi)j}c_{j,k,\theta}}{(s+\alpha_j-1)^k}\ , \tag{2.41}$$

Around $s=0$ one has $\frac{\sin(\pi s)}{\pi} = s + O(s^3)$ and hence

$$sg(s) = e^{i(\pi-\theta)s}(s^2+O(s^4))\left(\frac{c_0}{s^2} + \frac{c_1 + i(\theta-\pi)c_0 - \log\det{}_F\mathcal{S}}{s}\right) + s^2 p(s)\ , \tag{2.42}$$

where $p$ is meromorphic on $\mathbb{C}$ and holomorphic around $s=0$, giving the pole structure in (2.41) away from the origin. We therefore have near $s=0$

$$\frac{d}{ds}(sg(s)) = i(\pi-\theta)e^{i(\pi-\theta)s}\left(c_0 + s(c_1 + i(\theta-\pi)c_0 - \log\det{}_F\mathcal{S})\right)$$

$$+O(s^2) + e^{i(\pi-\theta)s}(c_1 + i(\theta-\pi)c_0 - \log\det{}_F\mathcal{S}) + O(s)\ .$$

And hence from (2.32)

$$\zeta_\theta'(A_1, A_2, 0) = c_1 - \log\det{}_F\mathcal{S} \tag{2.43}$$

and this is equation (2.28). □

**Remark 2.6.** *There is freedom in specifying* $\log\det_F \mathcal{S}_\lambda$ *up to the addition of a constant, and hence in specifying* $\mathcal{S}_\lambda$ *up to composition with an element of* $\mathrm{GL}_1(\mathcal{H}_\lambda) = \{\mathcal{E} \in GL(\mathcal{H}_\lambda) \mid \mathcal{E} - I \in L_1(\mathcal{H}_\lambda)\}$, *where* $L_1(\mathcal{H}_\lambda)$ *is the ideal of trace-class operators; that is,* $\mathcal{S}_\lambda\mathcal{E}$ *is also a scattering operator for any* $\mathcal{E} \in GL_1(\mathcal{H}_\lambda)$. *However, since* $\det_F : GL_1(\mathcal{H}_\lambda) \to \mathbb{C}^*$ *is a group homomorphism and the regularized limit is linear*

$$\mathrm{LIM}^{\theta}_{\lambda\to\infty}(g(\lambda) + c\,.\,f(\lambda)) = \mathrm{LIM}^{\theta}_{\lambda\to\infty}(g(\lambda)) + c\,.\,\mathrm{LIM}^{\theta}_{\lambda\to\infty}(f(\lambda)) \ , \tag{2.44}$$

*any constant* $c$, *then* (2.28) *and* (2.43) *are unambiguous.*

With the regularized limit $\mathrm{LIM}_{z\to 0}(h(z))$ denoting the constant term in the Laurent expansion of a function $h(z)$ around $z = 0$, we can recast (2.28) as follows:

**Proposition 2.7.** *If* $A_1, A_2$ *are* $\zeta$*-comparable, then*

$$\begin{aligned} \log\det_{\zeta,\theta}(A_1, A_2) &= -\mathrm{LIM}_{s\to 0}\left[\frac{i}{2\pi}\int_C \lambda^{-s-1}\log\det_F\mathcal{S}_\lambda \ d\lambda\Big|^{mer}\right] \\ &= -\left[\frac{i}{2\pi}\int_C \lambda^{-s-1}\log\det_F\mathcal{S}_\lambda \ d\lambda - \frac{\zeta_{rel,\theta}(0)}{s}\right]\Bigg|^{mer}_{s=0} \end{aligned} \tag{2.45}$$

*Equivalently,*

$$\begin{aligned} \log\det_{\zeta,\theta}(A_1, A_2) &= -\mathrm{LIM}_{s\to 0}\left[\Gamma(s)\zeta_\theta(A_1, A_2, s)|^{mer}\right] + \gamma\zeta_{rel,\theta}(0) \\ &= -\left[\Gamma(s)\zeta_\theta(A_1, A_2, s) - \frac{\zeta_{rel,\theta}(0)}{s}\right]\Bigg|^{mer}_{s=0} + \gamma\zeta_{rel,\theta}(0) \ . \end{aligned} \tag{2.46}$$

*Proof.* From (2.33),(2.42), around $s = 0$ one has

$$\begin{aligned} \frac{i}{2\pi}\int_C \lambda^{-s-1} \log\det_F\mathcal{S} \ d\lambda &= e^{i(\pi-\theta)s}\left(\frac{c_0}{s} + c_1 + i(\theta-\pi)c_0 - \log\det_F\mathcal{S} + O(s)\right) \\ &= \frac{c_0}{s} + c_1 - \log\det_F\mathcal{S} + O(s) \ , \end{aligned}$$

and hence (2.45) follows from (2.43).

On the other hand, $\Gamma(s) = s^{-1} + \gamma s + O(s)$ near $s = 0$, so from (2.31)

$$\begin{aligned} \Gamma(s)\zeta_\theta(A_1, A_2, s) &= \Gamma(s)sg(s) \\ &= e^{i(\pi-\theta)s}\left(\frac{c_0}{s} + c_1 + i(\theta-\pi)c_0 - \log\det_F\mathcal{S} + c_0\gamma + O(s)\right) \ , \end{aligned}$$

and so (2.46) follows similarly. □

We also have:

**Proposition 2.8.** *Let* $A_1, A_2$ *be* $\zeta$*-comparable and* $\zeta$*-admissible, then there is an asymptotic expansion as* $\mu \to \infty$ *in* $\Lambda_\theta$

$$\log\det_{\zeta,\theta}(A_1 - \mu) - \log\det_{\zeta,\theta}(A_2 - \mu) \sim$$

$$\sum_{\substack{j=0\\ j\neq J}}^{\infty}\sum_{k=0}^{1} b'_{j,k}(-\lambda)^{-\alpha_j+1}\log^k(-\lambda) + b_{J,0}\log(-\lambda) \ .$$

*In particular, the constant term is zero :* $\mathrm{LIM}^{\theta}_{\mu\to\infty}\log\left(\frac{\det_{\zeta,\theta}(A_1-\mu)}{\det_{\zeta,\theta}(A_2-\mu)}\right) = 0$ .

*Proof.* For $\mu\in\Lambda_\theta$ the operators $A_i-\mu$ are $\zeta$-comparable, and hence by Theorem 2.5

$$\log\det{}_{\zeta,\theta}(A_1-\mu, A_2-\mu) = \log\det{}_F\mathcal{S}_\mu - \mathrm{LIM}^{\theta}_{\lambda\to\infty}\log\det{}_F\mathcal{S}_{\mu+\lambda} \ .$$

Since $\mathrm{LIM}^{\theta}_{\lambda\to\infty}\log\det_F\mathcal{S}_{\mu+\lambda} = \mathrm{LIM}^{\theta}_{\lambda\to\infty}\log\det_F\mathcal{S}_{\lambda}$ the conclusion is reached from (2.26),(2.37). □

Finally, it is useful to note that a similar proof allows Theorem 2.5 to be abstracted and generalized slightly:

**Proposition 2.9.** *Let $\Phi$ be a function on $\mathbb{C}$ which is meromorphic at $0$ with Laurent expansion $\Phi(\lambda) = \sum_{j=-m}^{\infty} b_j(-\lambda)^j$, and holomorphic in a sector $\Lambda_\theta$ with for some $r\in\mathbf{Z}$ a uniform asymptotic expansion of $\partial\Phi/\partial\lambda$ as $\lambda\to\infty$ in $\Lambda_\theta$*

$$\frac{\partial\Phi}{\partial\lambda} \sim \sum_{j=-r}^{\infty}\sum_{k=0}^{1} a_{j,k}(\lambda)^{-\alpha_j}\log^k(-\lambda) \ , \tag{2.47}$$

*where $-\infty<\alpha_{-r}<\ldots<\alpha_{-1}\le 0<\alpha_0<\ldots<\alpha_j\nearrow\infty$. Then*

$$Z(s) = -\frac{i}{2\pi}\int_C \lambda^{-s}\frac{\partial\Phi}{\partial\lambda}\ d\lambda$$

*is holomorphic for $\mathrm{Re}(s)>1-\alpha_{-r}$ and has a meromorphic continuation to all of $\mathbb{C}$ with poles determined by the coefficients of (2.47). In particular, if $a_{J,1}=0$ (with $\alpha_J=1$) then $Z(s)$ is holomorphic around $s=0$ with $Z(0)=a_{J,0}$, and*

$$-Z'(0) = \Phi(0) - \mathrm{LIM}^{\theta}_{\lambda\to\infty}\Phi(\lambda) \ , \tag{2.48}$$

*where $\Phi(0) := \mathrm{LIM}_{\lambda\to 0}\Phi(\lambda) = b_0$.*

2.1. **Relative Heat Kernel Regularization.** In this Section we derive Theorem 2.5 using the heat operator trace for operators $A_1, A_2$ with spectrum contained in a sector of the right-half plane. This applies, for example, to the Dirac Laplacian on a manifold with boundary.

We assume that $R_\pi$ is a spectral cut for $A_1, A_2$, so that $\|(A_i-\lambda)^{-1}\| = O(|\lambda|^{-1})$ for large $\lambda$ in $\Lambda_\pi$ with $\delta>\pi/2$ in (2.1). Let $\mathcal{C}$ be a contour surrounding $\mathrm{sp}(A_1),\mathrm{sp}(A_2)$, coming in on a ray with argument in $(0,\pi/2)$, encircling the origin, and leaving on a ray with argument in $(-\pi/2,0)$. For $t>0$, one then has the heat operators

$$e^{-tA_i} := \frac{i}{2\pi}\int_{\mathcal{C}} e^{-t\lambda}(A_i-\lambda)^{-1}\ d\lambda = \frac{i}{2\pi}\int_{\mathcal{C}} t^{-m}e^{-t\lambda}\partial_\lambda^m(A_i-\lambda)^{-1}\ d\lambda \ .$$

If we assume $\partial_\lambda^m(A_i-\lambda)^{-1}$ is trace class for some $m$, then $e^{-tA_i}$ is trace class with

$$\mathrm{Tr}\,(e^{-tA_i}) = \frac{i}{2\pi}\int_{\mathcal{C}} t^{-m}e^{-t\lambda}\mathrm{Tr}\,(\partial_\lambda^m(A_i-\lambda)^{-1})\ d\lambda \ .$$

The resolvent trace expansions (2.5) thus imply heat trace expansions as $t \to 0$

$$\mathrm{Tr}\,(e^{-tA_i}) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{1} \tilde{a}_{j,k}^{(i)} t^{\alpha_j - 1}\log^k t \ , \tag{2.49}$$

with coefficients differing from those in (2.5) by universal constants, while $\mathrm{Tr}\,(e^{-tA_i}) = O(e^{-ct})$, some $c > 0$, as $t \to \infty$. The heat representation of the power operators

$$A_i^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-tA_i}\ dt \ , \qquad \mathrm{Re}(s) > 0 \ , \tag{2.50}$$

then implies $\zeta(A_i, s) = \Gamma(s)^{-1}\int_0^\infty t^{s-1}\mathrm{Tr}\,(e^{-tA_i})\ dt$ for $\mathrm{Re}(s) > 1 - \alpha_0$, with (2.49) giving the pole structure of the meromorphic extension to $\mathbb{C}$.

For positive $\zeta$-admissible operators the heat cut-off regularization, defined by

$$\log\det{}_{heat}(A_i) := \mathrm{LIM}_{\varepsilon\to 0}\int_\varepsilon^\infty -\frac{1}{t}\mathrm{Tr}\,(e^{-tA_i})\ dt \ , \tag{2.51}$$

picks out the constant term in the asymptotic expansion as $\varepsilon \to 0$ of

$$f(\varepsilon) := \int_\varepsilon^\infty -\frac{1}{t}\mathrm{Tr}\,(e^{-tA_i})\ dt \sim -\zeta(A_i,0)\log\varepsilon + \mathrm{LIM}_{\varepsilon\to 0}f(\varepsilon) - \sum_{j=0}^{\infty}\sum_{k=0}^{1} f_{j,k}\varepsilon^{\alpha_j-1}\log^k\varepsilon \ .$$

Since $\partial f/\partial\varepsilon = \varepsilon^{-1}\mathrm{Tr}\,(e^{-\varepsilon A_i})$, the existence of such an expansion follows from (2.49) and the small time asymptotics analogue of Lemma 2.4.

The definition in (2.51) is motivated by $\det_{heat}(A) = \det_F(A)$ in finite dimensions. However, if $H$ is infinite-dimensional and $A$ is determinant class, then $e^{-tA}$ is not trace class and (2.51) is undefined. There is, nevertheless, for any pair of $\zeta$-comparable operators $A_1, A_2$ a well-defined relative heat cut-off determinant

$$\log\det{}_{heat}(A_1, A_2) := \mathrm{LIM}_{\varepsilon\to 0}\int_\varepsilon^\infty -\frac{1}{t}\mathrm{Tr}\,(e^{-tA_1} - e^{-tA_2})\ dt \ . \tag{2.52}$$

This includes both when $A_1, A_2$ are determinant class or $\zeta$-admissible. In the former case, $e^{-tA_1} - e^{-tA_2}$ is now trace class, $\log\det_{heat}(A_1, A_2) = \int_0^\infty -\frac{1}{t}\mathrm{Tr}\,(e^{-tA_1} - e^{-tA_2})\ dt$ and $\det_{heat}(A_1, A_2) = \det_F(A_1)/\det_F(A_2)$. This is the $r$-integrated version of

$$\mathrm{Tr}\,(A_r^{-1}\dot{A}_r) = \int_0^\infty -t^{-1}\mathrm{Tr}\,(\frac{d}{dr}e^{-tA_r})\ dt \ , \qquad (A_r \ \text{ det class}) \tag{2.53}$$

where $A_1 \le A_r \le A_2$ is a smooth 1-parameter family of non-negative determinant class operators (to see (2.53), set $s = 1$ in (2.50) and use Duhamel's principle).

On the other hand, when the $A_i$ are $\zeta$-admissible

$$\det{}_{heat}(A_1, A_2) = \frac{\det_{heat}(A_1)}{\det_{heat}(A_2)} \ . \tag{2.54}$$

By (2.57), below, (2.54) is a restatement of (2.30) and (2.21).

The relative heat and $\zeta$-function $\zeta(A_1, A_2, s) = \Gamma(s)^{-1}\int_0^\infty t^{s-1}\mathrm{Tr}\,(e^{-tA_1} - e^{-tA_2})\ dt$ regularizations are related to the scattering determinant is as follows.

**Theorem 2.10.** *Let $A_1, A_2$ be positive $\zeta$-comparable operators with $\theta = \pi$, as above. Then as $\varepsilon \to 0$ there is an asymptotic expansion*

$$\int_\varepsilon^\infty -\frac{1}{t}\mathrm{Tr}\,(e^{-tA_1} - e^{-tA_2})\ dt \sim \log \det{}_F \mathcal{S} - \mathrm{LIM}_{\mu\to\infty} \log\det{}_F \mathcal{S}_{-\mu} - \zeta_{rel}(0)\Gamma'(1) - \zeta_{rel}(0)\log\varepsilon + \sum_{j=0}^{\infty}\sum_{k=0}^{1} c'_{j,k}\varepsilon^{\alpha_j - 1}\log^k \varepsilon\ . \tag{2.55}$$

*Hence*

$$\log \det{}_{heat}(A_1, A_2) = \log\det{}_F \mathcal{S} - \mathrm{LIM}_{\lambda\to\infty}\log\det{}_F\mathcal{S}_{-\lambda} - \zeta_{rel}(0)\Gamma'(1)\ . \tag{2.56}$$

*One has*

$$\det{}_\zeta(A_1, A_2) = \det{}_{heat}(A_1, A_2) e^{\zeta_{rel}(0)\Gamma'(1)}\ . \tag{2.57}$$

**Remark 2.11.** $\Gamma'(1) = -\gamma$.

*Proof.* Equation (2.57) follows from (2.56) and (2.28). Alternatively, it is proved directly, without reference to Theorem 2.5, by an obvious modification of the following proof of (2.55).

From (2.12), (2.1) we have

$$\mathrm{Tr}\,(e^{-tA_1} - e^{-tA_2}) = -\frac{i}{2\pi}\int_{\mathcal{C}} e^{-t\lambda}\frac{\partial}{\partial\lambda}\log\det{}_F\mathcal{S}_\lambda\ d\lambda = -t\frac{i}{2\pi}\int_{\mathcal{C}} e^{-t\lambda}\log\det{}_F\mathcal{S}_\lambda\ d\lambda\ . \tag{2.58}$$

Hence

$$\int_\varepsilon^\infty -\frac{1}{t}\mathrm{Tr}\,(e^{-tA_1} - e^{-tA_2})\ dt = \frac{i}{2\pi}\int_{\mathcal{C}}\lim_{\omega\to\infty}\left.\frac{e^{-t\lambda}}{-\lambda}\right|_{t=\varepsilon}^{\omega}\log\det{}_F\mathcal{S}_\lambda\ d\lambda$$

$$= \frac{i}{2\pi}\int_{\mathcal{C}}\lim_{\omega\to\infty}\frac{e^{-\mu}\log\det{}_F\mathcal{S}_{\mu/\omega}}{-\mu}\ d\mu - \frac{i}{2\pi}\int_{\mathcal{C}}\frac{e^{-\rho}\log\det{}_F\mathcal{S}_{\rho/\varepsilon}}{-\rho}\ d\rho\ , \tag{2.59}$$

using $\mu = \omega\lambda, \rho = \varepsilon\lambda$ and homotopy invariance to shift the contours $\omega\mathcal{C}, \varepsilon\mathcal{C}$ to $\mathcal{C}$.

Since $\lim_{\omega\to\infty}\log\det{}_F\mathcal{S}_{\mu/\omega} = \log\det{}_F\mathcal{S}$ and since the contour in the first term in (2.59) can be closed at $\infty$, we have

$$\frac{i}{2\pi}\int_{\mathcal{C}}\lim_{\omega\to\infty}\frac{e^{-\mu}\log\det{}_F\mathcal{S}_{\mu/\omega}}{-\mu}\ d\mu = \log\det{}_F\mathcal{S}\ . \tag{2.60}$$

Now from (2.26), (2.37), for any $\delta > 0$, as $\varepsilon \to 0$

$$\frac{\log\det{}_F\mathcal{S}_{\rho/\varepsilon}}{-\rho} = c_1(-\rho)^{-1} + c_0(-\rho)^{-1}\log(-\rho) + c_0(-\rho)^{-1}\log(\varepsilon) + \sum_{j=J+1}^{N-1}\sum_{k=0}^{1} c_{j,k}((-\rho)^{1-\alpha_j}\varepsilon^{1-\alpha_j}\left(\log(-\rho) + \log(\varepsilon)\right)^k + O(|\rho\varepsilon|^{-\alpha_N+\delta})\ . \tag{2.61}$$

Hence, as $\varepsilon \to 0$,

$$\frac{i}{2\pi}\int_{\mathcal{C}} \frac{e^{-\rho}\log \det{}_F \mathcal{S}_{\rho/\varepsilon}}{-\rho}\ d\rho = c_1 . \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\rho}(-\rho)^{-1}\ d\rho \tag{2.62}$$

$$+c_0 . \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\rho}(-\rho)^{-1}\log(-\rho)\ d\rho + c_0\log(\varepsilon) . \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\rho}(-\rho)^{-1}\ d\rho$$

$$\sum_{j=J+1}^{N-1}\sum_{k=0}^{1} c_{j,k}\varepsilon^{1-\alpha_j} . \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\rho}(-\rho)^{1-\alpha_j}\left(\log(-\rho)+\log(\varepsilon)\right)^k + O(|\varepsilon|^{\alpha_N+\delta})\ .$$

From the contour integral formula for the Gamma function

$$\Gamma(s)^{-1} = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\rho}(-\rho)^{-s}\ d\rho\ ,$$

we have

$$\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\rho}(-\rho)^{-1}\ d\rho = \Gamma(1)^{-1} = 1\ , \tag{2.63}$$

and

$$\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\rho}(-\rho)^{-1}\log(-\rho)\ d\rho\ = -\frac{d}{ds}_{|s=1}\left(\Gamma(s)^{-1}\right) = \Gamma'(1)\ . \tag{2.64}$$

From (2.59), (2.60), (2.62), (2.63), (2.64), we reach the conclusion. □

For closely related formulae see §3 of [21].

2.2. **Relative Eta Invariants.** The dependence of the relative $\zeta$-determinant on the choice of spectral cut $R_\theta$ is measured by the regularized limit in (2.28):

**Lemma 2.12.** *Let $A_1, A_2$ be $\zeta$-comparable operators for spectral cuts $\theta, \phi \in [0, 2\pi)$ with scattering matrices $\mathcal{S}^\theta_\lambda, \mathcal{S}^\phi_\mu$ chosen so that* $\det_F \mathcal{S}^\theta_0 = \det_F \mathcal{S}^\phi_0$. *Then*

$$\frac{\det_{\zeta,\theta}(A_1,A_2)}{\det_{\zeta,\phi}(A_1,A_2)} = \exp\left[-\mathrm{LIM}_{\alpha\to\infty}\log\left(\frac{\det_F \mathcal{S}^\theta_{e^{i\theta}\alpha}}{\det_F \mathcal{S}^\phi_{e^{i\phi}\alpha}}\right)\right]\ ,$$

*where* $\mathrm{LIM} = \mathrm{LIM}^0,\ \alpha \in \mathbb{R}_+$.

By Remark 2.6 the requirement $\det_F \mathcal{S}^\theta_0 = \det_F \mathcal{S}^\phi_0$ can always be fulfilled, and so (2.12) follows from Theorem 2.5 and (2.44). Notice that the exponent is defined only $\mod (2\pi i\mathbf{Z})$ due to the ambiguity in defining log, and that the Fredholm determinants are taken on $H_{e^{i\theta}\alpha}, H_{e^{i\phi}\alpha}$, respectively.

We consider this for self-adjoint $A_1, A_2$ with $\mathrm{sp}(A_i)\cap\mathbb{R}_\pm \neq \emptyset$; for example, for operators of Dirac-type. There are, then, up to homotopy, two choices for $\theta$:

$$\theta = \frac{\pi}{2}\ , \qquad -\frac{3\pi}{2} \leq \arg(\lambda) < \frac{\pi}{2}\ , \qquad (-1)^s = e^{-i\pi s}\ ,$$

or,

$$\theta = \frac{3\pi}{2}\ , \qquad -\frac{\pi}{2} \leq \arg(\lambda) < \frac{3\pi}{2}\ , \qquad (-1)^s = e^{i\pi s}\ ,$$

which we may indicate by $\theta = +, \theta = -$, respectively. We assume that $A_1, A_2$ are $\zeta$-comparable so that for $\mu \in \Lambda_\pm$

$$\mathrm{Tr}\,((A_1-\mu)^{-1} \;-\; (A_2-\mu)^{-1}) = -\frac{\partial}{\partial\mu}\log\det{}_F\mathcal{S}^{\pm}_{\mu} \tag{2.65}$$

$$\sim \;\sum_{j=0}^{\infty}\sum_{k=0}^{1} a^{\pm}_{j,k}(-\mu)^{-\alpha_j}\log^k(-\mu) \qquad \text{as } \; \mu\to\infty \; \text{ in } \Lambda_\pm \; . \tag{2.66}$$

We may omit the $\pm$ superscripts in the following, $\mu$ indicating the appropriate scattering operator.

Since taking conjugates switches between spectral cuts it is enough to assume $\zeta$-comparability for just one of $\theta = \pi/2$ or $3\pi/2$. Considering $\mu = \pm i\alpha \in \pm i\mathbb{R}_+$ gives

$$a^-_{J,0} = \overline{a^+_{J,0}} \tag{2.67}$$

and

$$\partial_\alpha \log\det{}_F\mathcal{S}_{-i\alpha} = \partial_\alpha\log\det{}_F\mathcal{S}^*_{i\alpha} \; . \tag{2.68}$$

Corresponding to $C_+ = C_{\pi/2}$ and $C_- = C_{3\pi/2}$ we have two relative $\zeta$-functions $\zeta_\pm(A_1, A_2, s)$. Their disparity at $s = 0$ is measured to first order by $\zeta(A_1^2, A_2^2, 0)$ and the *relative $\eta$-invariant*

$$\eta(A_1, A_2) := \mathrm{LIM}_{s\to 0}\,\eta(A_1, A_2, s)|^{mer} \; ,$$

where for $\mathrm{Re}(s) >> 0$,

$$\begin{aligned}\eta(A_1, A_2, s) &= \mathrm{Tr}\,(A_1|A_1|^{-s-1} - A_2|A_2|^{-s-1}) \\ &= \frac{i}{2\pi}\int_{C_\pi} \lambda^{-\frac{s+1}{2}}\mathrm{Tr}\,(A_1(A_1^2-\lambda)^{-1} - A_2(A_2^2-\lambda)^{-1})\;d\lambda \; ,\end{aligned} \tag{2.69}$$

and $C_\pi$ is a contour of type (2.4) with $\theta = \pi$. (Here $A_i^2 = A_i^* A_i$ have (dense) domains $\mathrm{dom}(A_i^2) = \{\xi \in H \mid \xi, A_i\xi \in \mathrm{dom}(A_i)\}$).

The existence of $\zeta_\pi(A_1^2, A_2^2, s)$ and $\eta(A_1, A_2, s)$ for $\mathrm{Re}(s) >> 0$, the justification of (2.69), and their meromorphic continuation to $\mathbb{C}$ follow from the $\zeta$-comparability of $A_1, A_2$ via the identities

$$A_i(A_i^2-\lambda)^{-1} = \frac{1}{2}\left[(A_i-\lambda^{1/2})^{-1} + (A_i+\lambda^{1/2})^{-1}\right] \; , \tag{2.70}$$

$$(A_i^2-\lambda)^{-1} = \frac{1}{2\lambda^{1/2}}\left[(A_i-\lambda^{1/2})^{-1} - (A_i+\lambda^{1/2})^{-1}\right] \; . \tag{2.71}$$

Here $\lambda^{1/2}$ is uniquely specified by $R_\pi$. It is important to observe that the transformation $\lambda \to \lambda^{1/2}$ opens $C_\pi$ out into a vertical contour

$$C_\pi \longmapsto C_{1/2} := C_{\frac{\pi}{2},\downarrow} \cup C_{-\frac{\pi}{2},\uparrow} \cup C_{\rho,\frac{\pi}{2},-\frac{\pi}{2}} \; . \tag{2.72}$$

From (2.70), $\|A_i(A_i^2-\lambda)^{-1}\| = O(|\lambda^{-1/2}|)$, so $A_i|A_i|^{-\frac{s+1}{2}}$ is defined for $\mathrm{Re}(s) > 0$. Since $A_1, A_2$ are $\zeta$-comparable, (2.70) implies $A_1(A_1^2-\lambda)^{-1} - A_2(A_2^2-\lambda)^{-1}$

is trace class, and, from (2.66), $\eta(A_1, A_2, s)$ is defined for $\mathrm{Re}(s) > 1 - \alpha_0$ (see Proposition 2.15).

If $A_1, A_2$ are individually $\zeta$-admissible then $\eta(A_i, s) = \mathrm{Tr}\,(A_i|A_i|^{-\frac{s+1}{2}})$ are defined, and since $\eta(A_1, A_2, s)$ and $\eta(A_1, s) - \eta(A_2, s)$ have the same pole structure

$$\eta(A_1, A_2, s) = \eta(A_1, s) - \eta(A_2, s) \qquad (A_i \ \ \zeta - \text{admissible}) \ . \tag{2.73}$$

Likewise, from (2.71), $\|(A_i^2 - \lambda)^{-1}\| = O(|\lambda^{-1}|)$ and $\zeta(A_1^2, A_2^2, s)$ is defined for $\mathrm{Re}(s) > 1 - \alpha_0$. More precisely, setting $\log = \log_\pi$ from here on, we have:

**Proposition 2.13.** *For* $\lambda \in \Lambda_\pi$

$$\mathrm{Tr}\,((A_1^2 - \lambda)^{-1} - (A_2^2 - \lambda)^{-1}) = -\frac{\partial}{\partial \lambda} \log \det{}_F \mathcal{S}_{\lambda^{1/2}} - \frac{\partial}{\partial \lambda} \log \det{}_F \mathcal{S}_{-\lambda^{1/2}} \ . \tag{2.74}$$

*With* $-\lambda = \alpha \in \mathbb{R}_+$

$$\mathrm{Tr}\,((A_1^2 + \alpha)^{-1} - (A_2^2 + \alpha)^{-1}) = \frac{\partial}{\partial \alpha} \log \det{}_F ((\mathcal{S}_{\pm i\sqrt{\alpha}})^* \mathcal{S}_{\pm i\sqrt{\alpha}}) \ . \tag{2.75}$$

*[This means either* $(\mathcal{S}_{-i\sqrt{\alpha}})^* \mathcal{S}_{-i\sqrt{\alpha}}$ *or* $(\mathcal{S}_{+i\sqrt{\alpha}})^* \mathcal{S}_{+i\sqrt{\alpha}}$*.] As* $\lambda \to \infty$ *in* $\Lambda_\pi$ *there is an asymptotic expansion*

$$\mathrm{Tr}\,((A_1^2 - \lambda)^{-1} - (A_2^2 - \lambda)^{-1}) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{1} a'_{j,k} (-\lambda)^{-\frac{\alpha_j + 1}{2}} \log^k(-\lambda) \ . \tag{2.76}$$

*In particular,*

$$a'_{J,0} = \frac{a^+_{J,0} + a^-_{J,0}}{2} \ , \qquad a'_{J,1} = 0 \qquad (\alpha_J = 1) \ . \tag{2.77}$$

*Proof.* The first equation is a consequence of (2.65), (2.71) and

$$\mathrm{Tr}\,((A_1^2 - \lambda)^{-1} - (A_2^2 - \lambda)^{-1}) = \Psi(\lambda^{1/2}) + \Psi(-\lambda^{1/2}) \ , \tag{2.78}$$

where $\Psi(\rho) = (2\rho)^{-1} \mathrm{Tr}\,((A_1 - \rho)^{-1} - (A_2 - \rho)^{-1})$. Here, one uses (2.72) in order to track which sector $\lambda^{1/2}$ is in, and hence whether $\mathcal{S}_{\lambda^{1/2}}$ is $\mathcal{S}^+_\mu$ or $\mathcal{S}^-_\mu$. The change of branch of log between (2.65) and (2.71) is unimportant.

If $-\lambda = \alpha \in \mathbb{R}_+$, so $\lambda \in C_{\pi,\uparrow}$, then $\lambda^{1/2} = -i\sqrt{\alpha}$ with respect to $R_\pi$. Since $\overline{\Psi(\rho)} = \Psi(\overline{\rho})$, the right-side of (2.78) becomes $\Psi(-i\sqrt{\alpha}) + \overline{\Psi(-i\sqrt{\alpha})}$, or, equivalently, $\overline{\Psi(i\sqrt{\alpha})} + \Psi(i\sqrt{\alpha})$, resulting in the $\pm$ alternatives in (2.75), which now follows similarly to (2.74) using $\overline{\partial_\alpha \log \det{}_F \mathcal{S}_{\pm\alpha^{1/2}}} = \partial_\alpha \log \det{}_F (\mathcal{S}_{\pm\alpha^{1/2}})^*$.

(2.76) follows from (2.66) and (2.78). It is built from both of the expansions (2.66) as $\lambda^{1/2} \to \infty$ in $\Lambda_\pm$, but the coefficients in (2.76) will not in general be of the simple form (2.77) due to the change in spectral cut. However, $\lambda^{-1}$ is unambiguously defined and (2.77) follows by comparing (2.66), (2.74), (2.76). $\square$

The content of Proposition 2.13 is that $A_1^2, A_2^2$ are $\zeta$-comparable:

**Theorem 2.14.** *Let $A_1, A_2$ be self-adjoint $\zeta$-comparable operators, as above. Then $\zeta(A_1^2, A_2^2, s)$ is regular around $s = 0$, and (with $\theta = \pi$, $\mathcal{S} = \mathcal{S}_0$)*

$$\begin{aligned} \det_\zeta(A_1^2, A_2^2) &= \det_F(\mathcal{S}^*\mathcal{S})e^{-\mathrm{LIM}_{\alpha\to\infty}\log\det_F((\mathcal{S}_{\pm i\alpha})^*\mathcal{S}_{\pm i\alpha})} \\ &= |\det_F\mathcal{S}|^2 e^{-\mathrm{LIM}_{\alpha\to\infty}\log|\det_F(\mathcal{S}_{\pm i\alpha})|^2} . \end{aligned} \tag{2.79}$$

*If $A_1, A_2$ are individually $\zeta$-admissible, then so are $A_1^2, A_2^2$ and, then,* $\det_\zeta(A_1^2, A_2^2) = \det_\zeta(A_1^2)/\det_\zeta(A_2^2)$.

*Proof.* The first statement is equation (2.77). From Proposition 2.9 with $\Phi(\lambda) = \log\det_F\mathcal{S}_{\lambda^{1/2}} + \log\det_F\mathcal{S}_{-\lambda^{1/2}}$ we obtain

$$\det_\zeta(A_1^2, A_2^2) = \det_F(\mathcal{S}^*\mathcal{S})e^{-\mathrm{LIM}^\pi_{\lambda\to\infty}\Phi(\lambda)} . \tag{2.80}$$

Notice, though $\log\det_F\mathcal{S}^\pm$ may differ by a constant, (2.80) is unambiguous. The regularized limit averages the limits in the sectors $\Lambda_\pm$ and so is well-defined in $\Lambda_\pi$. Equation (2.79) now follows from (2.75) and (2.80) by computing the limit along $R_- = -R_+$, and noting $\mathrm{LIM}_{\alpha\to\infty}f(\sqrt{\alpha}) = \mathrm{LIM}_{\alpha\to\infty}f(\alpha)$. The final statement follows on applying $\partial_\lambda^m$ to (2.71) for large $m$. □

The analogue of Proposition 2.13 for the eta-function is proved similarly:

**Proposition 2.15.** *$A_1, A_2$ are $\eta$-comparable, in so far as, for $\lambda \in \Lambda_\pi$,*

$$\mathrm{Tr}\,(A_1(A_1^2-\lambda)^{-1} - A_2(A_2^2-\lambda)^{-1}) = -\lambda^{1/2}\frac{\partial}{\partial\lambda}\log\det_F\mathcal{S}_{\lambda^{1/2}} + \lambda^{1/2}\frac{\partial}{\partial\lambda}\log\det_F\mathcal{S}_{-\lambda^{1/2}} , \tag{2.81}$$

*and as $\lambda \to \infty$ in $\Lambda_\pi$ there is an asymptotic expansion*

$$\mathrm{Tr}\,(A_1(A_1^2-\lambda)^{-1} - A_2(A_2^2-\lambda)^{-1}) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{1} a''_{j,k}(-\lambda)^{-\frac{\alpha_j}{2}}\log^k(-\lambda) . \tag{2.82}$$

From (2.82) we obtain the singularity structure

$$\Gamma(\frac{s+1}{2})\eta(A_1, A_2, s) \sim \sum_{\substack{j=0\\ j\neq J}}^{\infty}\sum_{k=0}^{1}\frac{2^k A_{j,k}}{(s+\alpha_j-1)^{k+1}} + \frac{2A_{J,0}}{s} + \frac{4A_{J,1}}{s^2} , \tag{2.83}$$

where the $A_{j,k}$ differ from the $a''_{j,k}$ by universal constants. Since $A_1, A_2$ are $\zeta$-comparable, from (2.81) we find that $a''_{J,1} = 0$, and $A_{J,1} = 0$. Hence, as $\Gamma(s)$ is regular at $s = 1/2$, $\eta(A_1, A_2, s)$ can have at most a simple pole at $s = 0$.

Though Proposition 2.13 and Proposition 2.15 are ostensibly similar, the $\eta$-invariant has a quite different character. In particular, it is not necessary for $A_i$ to be invertible in order to define $\eta(A_i), \eta(A_1, A_2)$ – we require only that at $\lambda = 0$ the resolvents

$(A_i - \lambda)^{-1}$ are meromorphic[2]. It is then convenient to consider

$$\widetilde{\eta}(A_1, A_2) = \frac{\eta(A_1, A_2) + \dim \mathrm{Ker}(A_1) - \dim \mathrm{Ker}(A_2)}{2} ,$$

since $\widetilde{\eta}(A_1, A_2) \mod (\mathbf{Z})$ varies smoothly with 1-parameter families [5, 14, 20, 16].

The topological nature of $\eta(A)$ originates in the following difference of regularized limits in the sectors $\Lambda_\pm$:

**Theorem 2.16.** *If $A_1, A_2$ are self-adjoint $\zeta$-comparable operators, as above, then*
(2.84)

$$\widetilde{\eta}(A_1, A_2) = \frac{1}{2\pi i}\mathrm{LIM}_{\alpha\to\infty} \left(\log \det{}_F \mathcal{S}_{-i\alpha} - \log \det{}_F \mathcal{S}_{i\alpha}\right) + \frac{1}{2}\zeta(A_1^2, A_2^2, 0) \qquad \mod (\mathbf{Z}).$$

*Proof.* With $\mu = \lambda^{1/2}$ we have from (2.81), (2.72) that

$$\eta(A_1, A_2, s) = \frac{i}{2\pi}\int_{C_{1/2}} \mu^{-s}[\mathrm{Tr}\,((A_1 - \mu)^{-1} - (A_2 - \mu)^{-1}) + \mathrm{Tr}\,((A_1 + \mu)^{-1} - (A_2 + \mu)^{-1})]\, d\mu$$

$$= \frac{i}{2\pi}\int_{\mathrm{Re}(s)=c} \mu^{-s} \left(\partial_\mu \log \det{}_F \mathcal{S}_{-\mu} - \partial_\mu \log \det{}_F \mathcal{S}_{\mu}\right)\, d\mu \, , \tag{2.85}$$

where $c$ is positive and less than the smallest positive spectral value of $A_1$ or $A_2$.

Since $\mu^{-s} \log \det_F \mathcal{S}_\mu \to 0$ for $\mathrm{Re}(s) > 1 - \alpha_0$ as $\mu \to \infty$, integrating (2.85) gives

$$\eta(A_1, A_2) = \mathrm{LIM}_{s\to 0}(sG(s)) \, , \tag{2.86}$$

where $G(s) = \frac{i}{2\pi}\int_{\mathrm{Re}(\mu)=c} \mu^{-s-1}\, F(\mu)\, d\mu$, and

$$F(\mu) = \log \det{}_F \mathcal{S}_{-\mu} - \log \det{}_F \mathcal{S}_{\mu} \qquad \mod (2\pi i\mathbf{Z}) \, . \tag{2.87}$$

Here, since

$$\int_{\mathrm{Re}(\mu)=c} \mu^{-s-1}\, d\mu = 0 \qquad\qquad \mathrm{Re}(s) > 0 \, , \tag{2.88}$$

the $\mod (2\pi i\mathbf{Z})$ ambiguity in (2.87) is not seen in $G(s)$.

At $\mu = 0$: though $r(\mu) = \mathrm{Tr}\,((A_1 - \mu)^{-1} - (A_2 - \mu)^{-1})$ is meromorphic with residue $-\dim \mathrm{Ker}(A_1) + \dim \mathrm{Ker}(A_2)$, $r(\mu) + r(-\mu) = \partial_\mu F(\mu)$ is regular, and hence $(-\mu)^{-1}F(\mu)$ is meromorphic with a Laurent expansion $(-\mu)^{-1}F(\mu) = \sum_{j=-1}^{\infty} b_j(-\mu)^j$. Let $F_0(\mu) = (-\mu)^{-1}(F(\mu) - b_{-1})$.

From (2.88) and since $F_0(\mu)$ is regular at $\mu = 0$

$$G(s) = \frac{i}{2\pi}\int_{\mathrm{Re}(\mu)=c} \mu^{-s}\, F_0(\mu)\, d\mu = \frac{i}{2\pi}\int_{i\mathbb{R}} \mu^{-s}\, F_0(\mu)\, d\mu \, .$$

[2]This has consequences topologically: concretely, the eta-invariant provides a canonical generator for $\pi_1(\mathcal{F}_{sa})$, where $\mathcal{F}_{sa}$ is the space of self-adjoint Fredholm operators, and a transgression form in the relative family's index [18, 27].

We have $\mu = re^{\pm i\pi/2}$ on $\pm i\mathbb{R}$ and since the orientation goes from $+i\infty$ to $-i\infty$,

$$G(s) = -\frac{e^{-\frac{i\pi s}{2}}}{2\pi}\int_{\infty}^{0} r^{-s}\, F_0(re^{\frac{i\pi s}{2}})\, dr + \frac{e^{\frac{i\pi s}{2}}}{2\pi}\int_{0}^{\infty} r^{-s}\, F_0(re^{\frac{-i\pi s}{2}})\, dr \ . \tag{2.89}$$

By the argument in Theorem 2.5, if $h$ is holomorphic in $\Lambda_\theta$ with an asymptotic expansion $h(\lambda) \sim h_1(-\lambda)^{-1} + h_0(-\lambda)^{-1}\log(-\lambda) + \sum_{j=0}^{\infty}\sum_{k=0}^{1} h_{j,k}(-\lambda)^{-\alpha_j}\log^k(-\lambda)$, with $0 < \alpha_j \nearrow \infty$, as $\lambda \to \infty$, then $h(s) = \int_0^\infty r^{-s} f(re^{i\theta})\, dr$ defined for $\mathrm{Re}(s) >> 0$ extends meromorphically to $\mathbb{C}$ with singularity structure around $s = 0$

$$h(s) = e^{i(\pi-\theta)}\left(\frac{h_0}{s^2} + \frac{h_1 + i(\theta-\pi)h_0}{s}\right) + p(s) \ , \tag{2.90}$$

where $p(s)$ is meromorphic on $\mathbb{C}$ with poles at $s = 1 - \alpha_j \neq 0$.

Since $A_1, A_2$ are $\zeta$-comparable, as in (2.37) we obtain asymptotic expansions

$$F_0(\mu) \sim a_1^{\pm}(-\mu)^{-1} + a_0^{\pm}(-\mu)^{-1}\log(-\mu) + \sum_{j=0}^{\infty}\sum_{k=0}^{1} c_{j,k}^{\pm}(-\lambda)^{-\alpha_j}\log^k(-\lambda) \ , \tag{2.91}$$

as $\mu \longrightarrow \pm i\infty$, where $0 < \alpha_j \nearrow \infty$. From (2.87) we compute

$$a_1^{\pm} = \mathrm{LIM}_{\mu\to\pm i\infty}\left(\log\det{}_F\mathcal{S}_\mu - \log\det{}_F\mathcal{S}_{-\mu}\right) - b_{-1} \mp i\pi a_{J,0}^{\pm} \qquad \mathrm{mod}\ (2\pi i\mathbf{Z}), \tag{2.92}$$

and

$$a_0^{\pm} = \pm(a_{J,0}^{\pm} + a_{J,0}^{\mp}) \ , \tag{2.93}$$

cf. (2.37) ( $i\pi a_{J,0}^{\pm}$ in (2.92) arises from $\log(\mu) = \log(-\mu) + i\pi$).

From (2.89), (2.90), (2.91)

$$\begin{aligned} sG(s) &= -\frac{e^{-\frac{i\pi s}{2}}}{2\pi i}\left(\frac{a_0^+}{s} + a_1^+ - \frac{\pi i}{2}a_0^+\right) + \frac{e^{\frac{i\pi s}{2}}}{2\pi i}\left(\frac{a_0^-}{s} + a_1^- - \frac{3\pi i}{2}a_0^-\right) + O(s) \\ &= \frac{(a_0^- - a_0^+)/2\pi i}{s} + \frac{(a_0^+ - a_0^-)}{2} + \frac{(a_1^- - a_1^+)}{2\pi i} + O(s) \ . \end{aligned} \tag{2.94}$$

Hence from (2.86), (2.92), (2.93)

$$\begin{aligned} \eta(A_1, A_2) &= \frac{(a_0^+ - a_0^-)}{2} + \frac{(a_1^- - a_1^+)}{2\pi i} \\ &= \frac{1}{\pi i}\mathrm{LIM}_{\alpha\to\infty}\left(\log\det{}_F\mathcal{S}_{-i\alpha} - \log\det{}_F\mathcal{S}_{i\alpha}\right) + \frac{a_{J,0}^+ + a_{J,0}^-}{2} \qquad \mathrm{mod}\ (2\mathbf{Z}) \\ &= \frac{1}{\pi i}\mathrm{LIM}_{\alpha\to\infty}\left(\log\det{}_F\mathcal{S}_{-i\alpha} - \log\det{}_F\mathcal{S}_{i\alpha}\right) + a_{J,0}^{'} \qquad \mathrm{mod}\ (2\mathbf{Z}), \end{aligned}$$

the final equality using (2.77). Since $\zeta(A_1^2, A_2^2, 0) = a_{J,0}^{'} - \dim\mathrm{Ker}(A_1^2) + \dim\mathrm{Ker}(A_1^2)$, and, by self-adjointness, $\dim\mathrm{Ker}(A_i^2) = \dim\mathrm{Ker}(A_i)$ we reach the conclusion. $\square$

**Remark 2.17.** *(1) Similar regularized winding numbers to* (2.84) *for suspended* $\psi$dos *have been studied in* [17].

*(2) The methods of § 2.1 can be used to obtain* (2.84) *through the heat formula*

$$\eta(A_1, A_2, s) = \Gamma(\frac{s+1}{2})^{-1} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{Tr}\,(A_1 e^{-tA_1^2} - A_2 e^{-tA_2^2})\, dt\ .$$

*(3) From* (2.68)*, the regularized limit in* (2.84) *is pure imaginary, while from* (2.67) *and* (2.77) $\zeta(A_1^2, A_2^2, 0)$ *is real. When* $A_1, A_2$ *are invertible this corresponds to the the role of these invariants in defining the phase of the determinant*

$$\det_{\zeta,\pm}(A_1, A_2) = e^{\mp i\frac{\pi}{2}(\eta(A_2,A_2)-\zeta(A_1^2,A_2^2,0))} \det_{\zeta,\pi}(|A_1|, |A_2|)\ . \tag{2.95}$$

2.3. **A Multiplicativity Property.** We refer to $\zeta$-comparable operators $A_1, A_2$ as *strongly $\zeta$-comparable* if $a_{j,k} = 0$ for $j \leq J$ in (2.13); in particular, $\zeta_{rel}(0) = 0$.

From (2.29), if $A_1, A_2$ are strongly $\zeta$-comparable then LIM = lim and

$$\det_{\zeta,\theta}(A_1, A_2) = \det_F \mathcal{S}\,.\,e^{-\lim^{\theta}_{\lambda\to\infty} \log\det_F \mathcal{S}_\lambda}\ . \tag{2.96}$$

More precisely, $\zeta_\theta(A_1, A_2, s)$ is then holomorphic for $\mathrm{Re}(s) > 1 - \alpha_{J+1}$ and so at $s = 0$, without continuation, and (2.96) follows from

$$\zeta_\theta(A_1, A_2, s) = -\frac{\sin(\pi s)}{\pi} e^{i(\pi-\theta)s} \int_0^\infty r^{-s} \frac{\partial}{\partial r} \log\det_F(\mathcal{S}_{re^{i\theta}})\, dr\ .$$

This applies to the following multiplicativity property:

**Theorem 2.18.** *Let* $A : H \to H$ *be closed and invertible with spectral cut* $R_\theta$ *with* $\|(A-\lambda)^{-1}\| = O(|\lambda|^{-1})$ *as* $\lambda \to \infty$ *in* $\Lambda_\theta$ *and let* $Q = I + W : H \to H$ *with* $W$ *of trace class. If the operator* $AW$ *is trace class, then* $(AQ, A)$ *are strongly $\zeta$-comparable and*

$$\det_{\zeta,\theta}(AQ, A) = \det_F Q\ . \tag{2.97}$$

*If* $A$ *is $\zeta$-admissible, then so is* $AQ$ *and*

$$\det_{\zeta,\theta}(AQ) = \det_{\zeta,\theta}(A).\det_F Q\ . \tag{2.98}$$

Let us point out some immediate consequences. First, we have:

**Proposition 2.19.** *Let* $M$ *be a closed $n$-manifold and* $\mathsf{D} : H^s(M, E) \to H^{s-d}(M, E)$ *an elliptic* $\psi$do *of order* $d > 0$ *acting on sections of a vector bundle* $E$ *over* $M$*. Let* $Q = I + W$ *where* $W$ *is a* $\psi$do *on* $L^2(M, E)$ *of order* $\mathrm{ord}(W) < -n - d$*. Then*

$$\det_{\zeta,\theta}(\mathsf{D}Q) = \det_{\zeta,\theta}(\mathsf{D}).\det_F Q\ ,$$

*In particular, this holds if* $W$ *is a smoothing operator.*

*Proof.* It is well known that in this case $\mathsf{D}$ is $\zeta$-admissible. On the other hand, $\mathrm{ord}(\mathsf{D}W) < -n$ and hence $\mathsf{D}W$ is trace class. $\square$

This generalizes Lemma(2.1) of [13]. On the other hand, using the multiplicativity of the Fredholm determinant, setting $A = Q_2$ and $Q = Q_2^{-1} Q_1$ we have:

**Proposition 2.20.** *Theorem 2.18,* (2.97)*, applies to any bounded operator $A$ on $H$. In particular, $(Q, I)$ are strongly $\zeta$-comparable, with $I$ the identity operator, and*

$$\det_{\zeta,\theta}(Q, I) = \det_F Q \ . \tag{2.99}$$

*Equivalently, if $Q_1, Q_2$ are determinant class, they are strongly $\zeta$-comparable and*

$$\det_{\zeta,\theta}(Q_1, Q_2) = \det_F(Q_2^{-1}Q_1) = \frac{\det_F Q_1}{\det_F Q_2} \ . \tag{2.100}$$

Thus, although $\zeta_\theta(Q_i, s)$ is undefined if $Q_i$ is of determinant class for any $s$ (with $H$ infinite-dimensional), $\zeta_\theta(Q_1, Q_2, s)$ is defined and holomorphic for $\mathrm{Re}(s) > -1$ (see below). Since the contour can be closed at $\infty$ this extends to all $s$—equivalently, (2.99) is independent of $\theta$, providing one perspective on the following.

First, note that for self-adjoint strongly $\zeta$-comparable operators, $\eta(A_1, A_2, s)$ is holomorphic for $\mathrm{Re}(s) > 1 - \alpha_{J+1}$, and (2.84) is immediate with LIM = lim on setting $s = 0$ in (2.85).

**Proposition 2.21.** *Let $Q_1, Q_2$ be self-adjoint and determinant class. Then $\eta(Q_1, Q_2, s)$ defined for $\mathrm{Re}(s) > -2$ extends to an entire function and*

$$\widetilde{\eta}(Q_1, Q_2) \in \mathbf{Z} \ .$$

*Proof.* The first statement follows from (2.102). The identity is immediate from (2.84) and (2.103). $\square$

An equivalent view point is that $\det_{\zeta,\theta}(Q_1, Q_2)$ is real by (2.100), and so the phase in (2.95) must be real. Matters are quite different for differential operators (§ 3.4).

The proof Theorem 2.18 is as follows.

*Proof.* We have $AQ - \lambda = A - \lambda + S$, where $S = AW$ is trace class. Hence $(AQ - \lambda)^{-1} - (A - \lambda)^{-1}$ is trace class, and for $\lambda$ large

$$(AQ - \lambda)^{-1} = (A - \lambda)^{-1} + \sum_{k\geq 1}(A - \lambda)^{-1}(S(A - \lambda)^{-1})^k \ . \tag{2.101}$$

From the trace norm estimate

$$\|(A - \lambda)^{-1}(S(A - \lambda)^{-1})^k\|_{Tr} \leq \|(A - \lambda)^{-1}\|^{k+1}(\|S\|_{Tr})^k < C|\lambda|^{-k-1} \ ,$$

as $\lambda \to \infty$, we have $\|(AQ - \lambda)^{-1} - (A - \lambda)^{-1}\|_{Tr} = O(|\lambda|^{-2})$ and hence that

$$|\mathrm{Tr}\ \left((AQ - \lambda)^{-1} - (A - \lambda)^{-1}\right)| = O(|\lambda|^{-2}) \ . \tag{2.102}$$

Therefore $\zeta_\theta(AQ, A, s) = \mathrm{Tr}\,((AQ)^{-s} - A^{-s})$ is holomorphic for $\mathrm{Re}(s) > -1$ and $\zeta_\theta(AQ, A, 0) = 0$. Using the symmetry of the trace we find

$$\mathrm{Tr}\ \left((AQ - \lambda)^{-1} - (A - \lambda)^{-1}\right) = -\frac{\partial}{\partial\lambda}\log\det_F((A - \lambda)^{-1}(AQ - \lambda)),$$

so the scattering matrix is $\mathcal{S}_\lambda = (A - \lambda)^{-1}(AQ - \lambda)$. Hence from (2.96)

$$\det_{\zeta,\theta}(AQ, A) = \det_F Q \,.\, e^{\lim^\theta_{\lambda\to\infty}\log\det_F \mathcal{S}_\lambda} \ .$$

Finally, it is easy to see that $\det_F \mathcal{S}_\lambda = 1 + O(|\lambda|^{-1})$ for large $\lambda$ and hence that

$$\lim^{\theta}_{\lambda\to\infty} \log \det_F \mathcal{S}_\lambda = 0 \quad \mathrm{mod}(2\pi i\mathbb{Z}), \tag{2.103}$$

proving (2.97).

If $A$ is $\zeta$-admissible, then applying $\partial_\lambda^m$ to (2.101) for large enough $m$, we find $\partial_\lambda^m (AQ-\lambda)^{-1}$ is trace class with an expansion (2.5) with no term $(-\lambda)^{-1}\log(-\lambda)$. Hence (2.98) follows from (2.30). $\square$

**Remark 2.22.** *In fact,* $\det_{\zeta,\theta}(AQ, A) = \det_F Q . e^{-\mathrm{LIM}^{\theta}_{\lambda\to\infty} \log\det_F \mathcal{S}_\lambda}$ *for any determinant class* $Q$*— though expected, the vanishing of the regularized limit is unresolved.*

## 3. An Application to Global Boundary Problems of Dirac-type

We turn now to the application of Theorem 2.5 to elliptic differential operators on manifolds with boundary.

3.1. **Analytic Preliminaries.** Let $X$ be a compact Riemannian manifold with (closed) boundary manifold $\partial X = Y$. Let $E^1, E^2$ be Hermitian vector bundles over $X$ and let $A : C^\infty(X, E^1) \to C^\infty(X, E^2)$ be a first-order elliptic differential operator. We assume a collar neighborhood $U = [0,1) \times Y$ of the boundary such that

$$A_{|U} = \sigma\left(\frac{\partial}{\partial u} + \mathcal{A} + R\right), \tag{3.1}$$

where $\mathcal{A}$ is a first-order self-adjoint elliptic operator on $C^\infty(Y, E^1_{|Y})$, $R$ is an operator of order 0, and $\sigma : E^1_{|U} \to E^2_{|U}$ a unitary bundle isomorphism constant in $u$. When (3.1) holds, then $A$ is of *Dirac-type*. The case when $R = 0$ is called the *product case*.

We define the space of interior solutions of $A$

$$\mathrm{Ker}(A, s) := \{\psi \in H^s(X, E^1) \mid A\psi = 0 \ \text{ in } \ X\backslash Y\}\ ,$$

and its restriction to $Y$

$$H(A, s) := \gamma_0 \mathrm{Ker}(A, s)\ ,$$

where, for each real $s > 1/2$, $\gamma_0 : H^s(X, E^1) \to H^{s-1/2}(Y, E^1_{|Y})$ is the continuous operator restricting sections of $E^1$ in the $s^{th}$ Sobolev completion to the boundary. Because of the *Unique Continuation Property* $\gamma_0 : \mathrm{Ker}(A, s) \to H(A, s)$ is a bijection, while the *Poisson operator*

$$\mathcal{K}_A := r\tilde{A}^{-1}\tilde{\gamma}^*\sigma : H^{s-1/2}(Y, E^1_{|Y}) \longrightarrow \mathrm{Ker}(A, s) \subset H^s(X, E^1)\ . \tag{3.2}$$

defines a canonical left inverse to $\gamma_0$. Here, using the doubling construction [4] for example, we consider $X$ as embedded in a closed manifold $\tilde{X}$ with Hermitian bundles $\tilde{E}_i$ such that $(\tilde{E}_i)_{|X} = E_i$, and such that $A$ extends to an invertible elliptic operator $\tilde{A} : H^s(\tilde{X}, \tilde{E}^1) \to H^{s-1}(\tilde{X}, \tilde{E}^2)$, and where $\tilde{\gamma} : H^s(\tilde{X}, \tilde{E}^1) \to H^{s-1/2}(Y, E^1_{|Y})$, $r : H^s(\tilde{X}, \tilde{E}^2) \to H^s(X, E^2)$ are the continuous restriction operators.

**Proposition 3.1.** [29, 31, 4, 9, 10] *The restriction*

$$P(A) := \tilde{\gamma}\mathcal{K}_A \tag{3.3}$$

*of $\mathcal{K}_A$ to $Y$ is a $\psi$do projection of order $0$ (the Calderon projection) on the space of boundary sections $H^{s-1/2}(Y, E^1_{|Y})$ with range $H(A,s)$.*

*For $(y,\xi) \in T^*Y\backslash\{0\}$, the principal symbol $\sigma[P(A)](y,\xi) : E^1_y \to E^1_y$ is the orthogonal projection with range $N_+(y,\xi)$ equal to the direct sum of eigenspaces of the principal symbol of $\mathcal{A}$ with positive eigenvalue. Therefore $\sigma(P(A))$ is independent of the operator $R$.*

*In general, $P(A)$ is only a projector (an indempotent), but*

$$P(A)_{\rm ort} := P(A)^*P(A)(P(A)P(A)^* + (I - P(A)^*)(I - P(A)))^{-1}, \tag{3.4}$$

*is the $\psi$do projection (unique self-adjoint indempotent) with range*

$$\mathrm{ran}\,(P(A)_{\rm ort}) = \mathrm{ran}\,(P(A)) = \mathrm{ran}\,(P(A)^*)\ . \tag{3.5}$$

*and principal symbol $\sigma[P(A)] = \sigma[P(A)_{\rm ort}]$ .*

The Calderon projection provides a natural basepoint with which to define global boundary problems:

**Definition 3.2.** [31, 10] *A classical $\psi$do $B$ of order $0$ acting in $H^s(Y, E^1_{|Y})$ with principal symbol $\sigma[B]$ defines a boundary condition for $A$ which is well-posed if:*

*(i) $B$ has closed range for each real $s$;*

*(ii) for $(y,\xi) \in T^*Y\backslash\ \{0\}$, $\sigma[B](y,\xi)$ maps $N_+(y,\xi)$ injectively onto the range of $\sigma[B](y,\xi)$ in $\mathbb{C}^N$.*

**Definition 3.3.** *A well-posed boundary condition $B$ for $A$ is admissible if the $\psi$do $B - P(A)$ is a $\psi$do of order $< -n$.*

**Remark 3.4.** *In the following we shall for clarity and brevity assume that if $B$ is admissible then $B - P(A)$ is a smoothing operator. The modifications needed for the general case are straightforward.*

For each well-posed boundary condition $B$ for $A$ the global boundary problem

$$A_B = A : \mathrm{dom}\,(A_B) \to L^2(X, E^2) \tag{3.6}$$

with domain

$$\mathrm{dom}\,(A_B) = \{\psi \in H^1(X, E^1) \mid B\gamma_0\psi = 0\}$$

is a closed operator from $L^2(X, E^1)$ to $L^2(X, E^2)$. An equivalent global boundary problem is obtained by replacing $B$ by the $\psi$do projection $P[B] := P_{\mathrm{Ker}(B)^\perp}$ so that

$$A_B = A_{P[B]}\ , \tag{3.7}$$

where for any closed subspace $W \subset H_Y$, $P_W$ denotes the (orthogonal) projection with range $W$.

The following preferred sub-class of well-posed boundary conditions is of special interest. By an *APS-type boundary condition* we mean a $\psi$do projection $P$ of order 0 on $H_Y$ such that $P(A) - P$ is a $\psi$do of order $-1$. The *pseudodifferential Grassmannian* $Gr_1(A)$ is the infinite-dimensional manifold parameterizing such projections, each such $P \in Gr_1(A)$ defines a global boundary problem $A_P : \mathrm{dom}\,(A_P) \longrightarrow L^2(X, E^2)$. In particular, $Gr_1(A)$ contains the APS projection $\Pi_{\geq}$. This property is quite crude in so far as it follows trivially from the equality $\sigma[P(A)] = \sigma[\Pi_{\geq}]$. If $R = 0$, the flow over the collar leads to the following harder result:

**Proposition 3.5.** [24, 10] *In the product case*

$$P(A) - \Pi_{\geq} \qquad \text{and} \qquad P(A)^* - \Pi_{\geq}$$

*are* $\psi$dos *of order* $-\infty$ *(smoothing operators).*

Tailored to the product case we therefore also consider the dense submanifold $Gr_\infty(A)$ of $Gr_1(A)$ parameterizing those $P$ such that $P - P(A)$ is a smoothing operator.

Clearly, any $P \in Gr_\infty(A)$ is admissible. The following facts will be useful later:

**Lemma 3.6.** *Let $B_1, B_2$ be admissible boundary conditions for $A$. Then each of $B_1 - B_2$, $P[B_1] - P[B_2]$, $B_1 P[B_2]^\perp$ are smoothing operators, where $P^\perp := I - P$, any projection $P$. In particular, if $B$ is admissible, then*

$$P[B] \in Gr_\infty(A)\ .$$

*Proof.* See Remark 3.4. The first statement is obvious from Definition 3.3. The second follows easily from $P[B_i] = (i/2\pi)\int_\Gamma (B_i^* B_i - \lambda)^{-1}\,d\lambda$, with $\Gamma$ a contour surrounding the origin and not enclosing any eigenvalues of $B_i^* B_i$. For the third, one has $B_1 P[B_2]^\perp = B_1(P[B_2]^\perp - P_{\mathrm{Ker}(B_1)}) = B_1(P[B_2]^\perp - P[B_1]^\perp)$. □

The existence of the Poisson operator reduces the construction of a parametrix for $A_B$ to the construction of a parametrix for the operator on boundary sections

$$S_A(B) = B \circ P(A) : H(A) \longrightarrow W = \mathrm{ran}\,(B)\ . \tag{3.8}$$

$S(B) = S_A(B)$ is a Fredholm operator with kernel and cokernel consisting of smooth sections. The corresponding properties for $A_B$ follow from canonical isomorphisms

$$\mathrm{Ker}(S(B)) \cong \mathrm{Ker}(A_B)\ , \qquad\qquad \mathrm{Coker}(S(B)) \cong \mathrm{Coker}(A_B)\ . \tag{3.9}$$

The first is defined by the Poisson operator. The second follows in the same way from $\mathrm{Coker}(A_B) = \mathrm{Ker}(A_B^*)$ and $\mathrm{Coker}(S(B)) = \mathrm{Ker}(S_{A^*}(B^\star))$. Here, the operator

$$A_B^* := (A_B)^* = A_{B^\star}^* : \mathrm{dom}\,(A_{B^\star}^*) \to L^2(X, E^1)$$

is the adjoint realization of $A_B$ with $\mathrm{dom}\,(A_B^*) = \{\phi \in H^1(X, E^2) \mid B^\star \gamma_0 \phi = 0\}$. $A^*$ is the formal adjoint of $A$ and the adjoint boundary condition on $L^2(Y, E^2_{|Y})$ is

$$B^\star = \sigma P[B]^\perp \sigma^{-1}\ . \tag{3.10}$$

This follows from Green's formula

$$< A\psi, \phi >_2 - < \psi, A^*\phi >_1 = - < \sigma\gamma_0\psi, \gamma_0\phi >_Y \ , \tag{3.11}$$

which takes the distributional form on $H^s(X, E^i)$

$$A^{-1}_{P(A)}A = I - \mathcal{K}_A\gamma_0 \ . \tag{3.12}$$

In the collar neighborhood $U$ of $Y$

$$A^*_{|U} = -\sigma^{-1}\left(\frac{\partial}{\partial u} + \sigma\mathcal{A}\sigma^{-1} + \sigma R\sigma^{-1}\right) \ . \tag{3.13}$$

Hence $A^*$ is of Dirac-type with Poisson operator $\mathcal{K}_{A^*} : H^{s-1/2}(Y, E^2_{|Y}) \to H^s(X, E^2)$ and Calderon projection $P(A^*) = \gamma_0\mathcal{K}_{A^*}$ having range $H(A^*) = \gamma_0\mathrm{Ker}(A^*)$. There is an obvious diffeomorphism $Gr_\infty(A^*) \cong Gr_\infty(A)$, $P^\star \leftrightarrow P$, which, in view of

$$P(A^*) = \sigma P(A)^\perp\sigma^{-1} = P(A)^* \ , \qquad H(A^*) = \sigma(H(A)^\perp) \ , \tag{3.14}$$

is base point preserving. As in (3.8), $A^*_B$ is modelled by the boundary operator

$$S_{A^*}(B^\star) = B^\star \circ P(A^*) = \sigma B^\perp P(A)^\perp\sigma^{-1} : \sigma H(A)^\perp \longrightarrow \sigma W^\perp \ , \tag{3.15}$$

where $W^\perp := \mathrm{ran}\,(P[B]^\perp)$.

**Remark 3.7.** *With the identifications* (3.9) *at hand, elementary arguments* [4] *yield the identities* (1.2) *and* (1.3). *For an alternative proof using functorial methods see* [27]. *Details on the above facts can be accessed in* [29, 31, 9, 10, 4, 28].

**• Construction of a relative inverse from $S(B)^{-1}$**

From (3.9), if $A_B$ is invertible then so is $S(B)$ and we can define the *Poisson operator of the global boundary problem* $A_B$ by

$$\mathcal{K}_A(B) := \mathcal{K}_A S(B)^{-1}P_W : H^{s-1/2}(Y, E^1_{|Y}) \longrightarrow H^s(X, E^1) \ ,$$

where $W = \mathrm{ran}\,(B)$. This restricts to an isomorphism

$$\mathcal{K}_A(B)_{|W} : W \longrightarrow \mathrm{Ker}(A) \tag{3.16}$$

with inverse

$$\left(\mathcal{K}_A(B)_{|W}\right)^{-1} = (B\gamma_0)_{|\mathrm{Ker}(A)} \ . \tag{3.17}$$

**Proposition 3.8.** *Let $B, B_1, B_2$ be well-posed for $A$ such that the global boundary problems $A_B, A_{B_1}, A_{B_1}$ are invertible. Then one has*

$$A_B^{-1}A = I - \mathcal{K}_A(B)B\gamma_0 \ , \tag{3.18}$$

*and hence*

$$A^{-1}_{B_1} = A^{-1}_{B_2} - \mathcal{K}_A(B_1)B_1\gamma A^{-1}_{B_2} \ . \tag{3.19}$$

*Proof.* From (3.12)

$$A_{P(A)}^{-1} = A_{P(A)}^{-1} A A_B^{-1} = (A_{P(A)}^{-1} A) A_B^{-1} = (I - \mathcal{K}\gamma_0) A_B^{-1} = A_B^{-1} - \mathcal{K}_A \gamma_0 A_B^{-1} \ . \tag{3.20}$$

And so

$$B\gamma_0 A_{P(A)}^{-1} = -B\gamma_0 \mathcal{K}_A \gamma_0 A_B^{-1} = -S(B)P(A)\gamma_0 A_B^{-1} \ .$$

Applying $\mathcal{K}_A(B)$ to both sides, we have $\mathcal{K}_A(B)B\gamma_0 A_{P(A)}^{-1} = -\mathcal{K}_A\gamma_0 A_B^{-1}$. Substituting in (3.20) yields

$$A_B^{-1} = A_{P(A)}^{-1} - \mathcal{K}_A(B)B\gamma_0 A_{P(A)}^{-1} \ . \tag{3.21}$$

Hence, since $(\mathcal{K}_A(B)B\gamma_0)\mathcal{K}_A\gamma_0 = \mathcal{K}_A\gamma_0$,

$$A_B^{-1}A = (I - \mathcal{K}_A\gamma_0) - \mathcal{K}_A(B)B\gamma_0(I - \mathcal{K}_A\gamma_0) = I - \mathcal{K}_A(B)B\gamma_0 \ .$$

Hence

$$A_{B_1}^{-1} = A_{B_1}^{-1} A_{B_2} A_{B_2}^{-1} = (A_{B_1}^{-1} A) A_{B_2}^{-1} = A_{B_2}^{-1} - \mathcal{K}_A(B_1) B_1 \gamma A_{B_2}^{-1} \ .$$

□

**Remark 3.9.** *The relative-inverse formula appears in various forms in the literature. We refer in particular to* [8, 10, 28].

3.2. **The Relative Abstract Determinant.** The scattering determinant for $\zeta$-comparable global boundary problems arises canonically at the level of determinant lines.

• **Determinant lines**

The determinant of a Fredholm operator $E : H^1 \to H^2$ exists abstractly not as a number but as an element $\det E$ of a complex line $\mathrm{Det}\,(E)$. Elements of the *determinant line* $\mathrm{Det}\,(E)$ are equivalence classes $[\mathcal{E}, \lambda]$ of pairs $(\mathcal{E}, \lambda)$, where $\mathcal{E} : H^1 \to H^2$ such that $\mathcal{E} - E$ is trace class[3] and relative to the equivalence relation $(\mathcal{E}q, \lambda) \sim (\mathcal{E}, \det_F(q)\lambda)$ for $q : H^1 \to H^1$ of determinant class. Complex multiplication on $\mathrm{Det}\,(E)$ is defined by

$$\mu.[\mathcal{E}, \lambda] = [\mathcal{E}, \mu\lambda] \ . \tag{3.22}$$

The *abstract determinant* $\det E := [E, 1]$ is non-zero if and only if $E$ is invertible, and there is a canonical isomorphism

$$\mathrm{Det}\,(E) \cong \wedge^{max}\mathrm{Ker}(E)^* \otimes \wedge^{max}\mathrm{Coker}(E) \ . \tag{3.23}$$

(Clearly, any two complex lines are isomorphic, the issue, here and below, is whether there is a canonical choice of isomorphism.)

Taking quotients of abstract determinants in $\mathrm{Det}\,(E)$ coincides with the (relative) Fredholm determinant:

[3] A bounded operator $T : H^1 \to H^2$ is trace class if $\|T\|_{Tr} := \mathrm{Tr}\,(T^*T)^{1/2} < \infty$, where here $Tr$ means the sum of the eigenvalues, but $T$ does not have a trace unless $H^1 = H^2$.

**Lemma 3.10.** *Let $E_1 : H^1 \to H^2, E_2 : H^1 \to H^2$ be Fredholm operators such that $E_i - E$ are trace class. Then provided $E_2$ is invertible*

$$\frac{\det(E_1)}{\det(E_2)} = \det{}_F(E_2^{-1}E_1) \ , \tag{3.24}$$

*where the quotient on the left side is taken in* $\mathrm{Det}\,(E)$.

*Proof.* The left side of (3.24) is the ratio

$$\frac{[E_1, 1]}{[E_2, 1]} = \frac{[E_2.(E_2^{-1}E_1), 1]}{[E_2, 1]} = \frac{[E_2, \det{}_F(E_2^{-1}E_1)]}{[E_2, 1]}$$

which from (3.22) is equal to the asserted determinant. □

For example, in Proposition 2.19, one has $\det_{\zeta,\theta}(\mathsf{D}Q)/\det_{\zeta,\theta}(\mathsf{D}) = \det(\mathsf{D}Q)/\det(\mathsf{D})$.

• **Relative determinant lines for global boundary problems**

For well-posed boundary conditions $B_1, B_2$ for $A$, the global boundary problems $A_{B_1}, A_{B_2}$ have different domains and hence the abstract determinants live in different complex lines

$$\det(A_{B_1}) \in \mathrm{Det}\,(A_{B_1}) \ , \qquad \det(A_{B_2}) \in \mathrm{Det}\,(A_{B_2}) \ .$$

(We assume here that the $A_{B_i}$ are invertible.) This means that the *relative abstract determinant*

$$\det(A_{B_1}, A_{B_2}) := \det(A_{B_1})/\det(A_{B_2})$$

is undefined as a complex number. (Equivalently, although of the form identity plus a smoothing operator, the operators $D_{B_1}^{-1}D_{B_2}$ and $D_{B_1}D_{B_2}^{-1}$ do not have Fredholm determinants). Rather $\det(A_{B_1}, A_{B_2})$ is a canonical element of the

$$\textit{relative determinant line} := \mathrm{Det}\,(B_2, B_1)$$

of the boundary Fredholm operator[4]

$$(B_2, B_1) := B_1 \circ P[B_2] : \mathrm{ran}\,(B_2) \longrightarrow \mathrm{ran}\,(B_1) \ .$$

More precisely, there is a canonical isomorphism

$$\mathrm{Det}\,(A_{B_1}) \cong \mathrm{Det}\,(A_{B_2}) \otimes \mathrm{Det}\,(B_2, B_1) \tag{3.25}$$

and a canonical isomorphism

$$\mathrm{Det}\,(A_B) \cong \mathrm{Det}\,(S(B)) \ , \qquad \det(A_B) \longleftrightarrow \det(S(B)) \ . \tag{3.26}$$

Formally, these follow from (3.9) and (3.23) and are the determinant line analogues of (1.2) and (1.3), for precise constructions see [25, 27]. The isomorphism (3.25) says that to define $\det(A_{B_1}, A_{B_2})$ as a complex number requires a non-zero element of $\mathrm{Det}\,(B_2, B_1)$. By an *auxiliary operator* for $A_{B_1}, A_{B_2}$ we mean an invertible operator $\mathcal{E} : \mathrm{ran}\,(B_2) \to \mathrm{ran}\,(B_1)$ such that $\mathcal{E} - (B_2, B_1)$ is trace-class. Such an operator

[4]This is the origin of gauge anomalies on manifolds with boundary.

defines the non-zero element $\det(\mathcal{E}) \in \mathrm{Det}\,(B_2, B_1)$ and we hence obtain a regularized relative determinant, defined as the quotient taken in $\mathrm{Det}\,(A_{B_1})$ via (3.25)

$$\det_{\mathcal{E}}(A_{B_1}, A_{B_2}) = \frac{\det A_{B_1}}{\det A_{B_2} \otimes \det(\mathcal{E})} \ .$$

In particular, if $(B_2, B_1)$ is invertible then we obtain the *Relative Canonical Determinant* $\det_{\mathcal{C}}(A_{B_1}, A_{B_2}) := \det_{(B_2,B_1)}(A_{B_1}, A_{B_2})$.

**Proposition 3.11.** *Let $\mathcal{E} = \mathcal{E}(B_1, B_2)$ be an auxiliary operator for the global boundary problems $A_{B_1}, A_{B_2}$. If $B_1, B_2$ are admissible boundary conditions, then*

$$\det_{\mathcal{E}}(A_{B_1}, A_{B_2}) = \det{}_F\left(\frac{S(B_1)}{\mathcal{E}S(B_2)}\right), \tag{3.27}$$

*where the Fredholm determinant is taken on $H(A)$, and the operator quotient means $(\mathcal{E}S(B_2))^{-1}\,S(B_1) : H(A) \longrightarrow H(A)$. For two choices of auxiliary operator $\mathcal{E}, \mathcal{E}'$, one has*

$$\det_{\mathcal{E}}(A_{B_1}, A_{B_2}) = \det{}_F(\mathcal{E}^{-1}\mathcal{E}').\det_{\mathcal{E}'}(A_{B_1}, A_{B_2}) \ . \tag{3.28}$$

*If $(B_2, B_1)$ is invertible, one has*

$$\det_{\mathcal{C}}(A_{B_1}, A_{B_2}) = \det{}_F\left(\frac{S(B_1)}{B_1 S(B_2)}\right) \ . \tag{3.29}$$

*Proof.* The identities (3.28) and (3.29) are obvious from (3.27). From

$$S(B_1) = \mathcal{E}B_2P(A) + (B_1P(A) - \mathcal{E}B_2P(A)) \ ,$$

and since $B_i - P(A)$ have smooth kernels and $\mathcal{E} - (B_2, B_1)$ is trace-class, it is readily verified that $(\mathcal{E}S(B_2))^{-1}\,S(B_1)$ is of determinant class on $H(A)$.

From (3.25), (3.26) we obtain a commutative diagram of canonical isomorphisms

$$\begin{array}{ccc}
\mathrm{Det}\,(A_{B_1}) & \xrightarrow{\simeq} & \mathrm{Det}\,(A_{B_1}) \otimes \mathrm{Det}\,(B_2, B_1) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{Det}\,(S(B_1)) & \xrightarrow{\simeq} & \mathrm{Det}\,(S(B_2)) \otimes \mathrm{Det}\,(B_2, B_1)
\end{array}$$

in which the vertical maps take the abstract determinant elements to each other, while in the bottom map $\det(\mathcal{E}S(B_2)) \longleftrightarrow \det(S(B_2)) \otimes \det \mathcal{E}$. (See [25].) By construction, we therefore have

$$\frac{\det A_{B_1}}{\det A_{B_2} \otimes \det(\mathcal{E})} = \frac{\det(S(B_1))}{\det(\mathcal{E}S(B_2))} \ ,$$

and by (3.24) this is the right side of (3.27). $\square$

3.3. **Relative $\zeta$-Determinant of First-Order Global Boundary Problems.** To see that $\det_{\mathcal{E}}((A-\lambda)_{B_1},(A-\lambda)_{B_2})$ is a scattering determinant for $A_{B_1}, A_{B_2}$ and to compute the regularized limit, we study the zeta determinant under variation of the operator and the boundary conditions.

First, we study the operator variation, with fixed boundary conditions:

**Proposition 3.12.** *Let $A_z : C^\infty(X,E^1)\to C^\infty(X,E^2)$ be a 1-parameter family of Dirac-type operators depending smoothly on a complex parameter $z$ such that in the collar $U$*

$$A_{z|U} = \sigma\left(\frac{\partial}{\partial u} + \mathcal{A}_z + R_z\right) \ , \tag{3.30}$$

*where $\sigma$ and the principal symbol $\sigma(\mathcal{A}_z)$ of $\mathcal{A}_z$ are independent of $z$.*

*Let $\dot{A}_z = (d/dz)A_z$ and let $B_1, B_2$ be admissible for $A_z$ such that $A_{z,B_1}, A_{z,B_2}$ are invertible for each $z$. Then $\dot{A}_z(A_{z,B_1}^{-1} - A_{z,B_2}^{-1})$ is a trace class operator on $L^2(X,E_2)$ with*

$$\mathrm{Tr}\,(\dot{A}_z(A_{z,B_1}^{-1} - A_{z,B_2}^{-1})) = \mathrm{Tr}\,\left(S_z(B_1)^{-1}d_zS_z(B_1) - S_z(B_2)^{-1}d_zS_z(B_2)\right) \ , \tag{3.31}$$

*where $S_z := S_{A_z}$ and $d_z$ is defined by the covariant derivative $P(A_z)\cdot\frac{d}{dz}\cdot P(A_z)$ on $\mathcal{H} = \cup_z H(A_z)$ (see Remark 2.2). Relative to a choice of auxiliary operator $\mathcal{E} : \mathrm{ran}\,(B_2)\to \mathrm{ran}\,(B_1)$ one has*

$$\mathrm{Tr}\,(\dot{A}_z(A_{z,B_1}^{-1} - A_{z,B_2}^{-1})) = \frac{d}{dz}\log\mathrm{det}_F\left(\frac{S_z(B_1)}{\mathcal{E}S_z(B_2)}\right) \ . \tag{3.32}$$

$\mathrm{Tr}$ *and* $\det_F$ *on the right-side of* (3.31), (3.32) *are taken on $H(A_z)$*

*Proof.* From (3.19) and (3.17) we compute that on $\mathrm{dom}(A_{z,B_1})$

$$\begin{aligned} B_1\gamma_0A_{z,B_2}^{-1}A_{z,B_1} &= B_1\gamma_0(A_{z,B_1}^{-1} - \mathcal{K}_z(B_2)\gamma_0A_{z,B_1}^{-1})A_{z,B_1}\\ &= -B_1\gamma_0\mathcal{K}_z(B_2)\gamma_0\\ &= -\mathcal{K}_z(B_1)^{-1}\mathcal{K}_z(B_2)B_2\gamma_0\\ &= -S_z(B_1)S_z(B_2)^{-1}B_2\gamma_0 \ . \end{aligned} \tag{3.33}$$

The vector bundle structure on $\cup_zH(A_z)\cong\cup_z\mathrm{Ker}(A_z)$ follows from the smooth dependence of the operators on $z$ [27]. Let $d_z$ be the induced operator covariant derivative. Since $\mathcal{K}_z(B_1)$ has range in $\mathrm{Ker}(A_z)$ then $A_z\mathcal{K}_z(B_1)=0$, and hence $\dot{A}_z.\mathcal{K}_z(B_1) = -A_z\dot{\mathcal{K}}_z(B_1)$. Since $B_1\gamma_0\mathcal{K}_z(B_1) = B_1\gamma_0$, then $B_1\gamma_0d_z(\mathcal{K}_z(B_1)) = d_z(B_1\gamma_0) = 0$ so that

$$\frac{d}{dz}(A_z).\mathcal{K}_z(B_1) = -A_{z,B_1}d_z\mathcal{K}_z(B_1) \ , \tag{3.34}$$

We shall also need the identity

$$
\begin{aligned}
B_2 d_z \gamma_0 \mathcal{K}_z(B_1) &= \frac{d}{dz}\,(B_2\gamma_0\mathcal{K}_z(B_1)) \\
&= \frac{d}{dz}\left(B_2 P(A_z) S_z(B_1)^{-1}B_1\right) \\
&= \frac{d}{dz}\left(S_z(B_2)S_z(B_1)^{-1}B_1\right)\ . \qquad (3.35)
\end{aligned}
$$

From Lemma 3.6 and (3.19) we have that

$$
A^{-1}_{z,B_1} - A^{-1}_{z,B_2} = -\mathcal{K}_z(B_1)B_1\gamma_0 A^{-1}_{z,B_2} = -\mathcal{K}_z(B_1)B_1 P[B_2]^{\perp}\gamma_0 A^{-1}_{z,B_2}\ , \qquad (3.36)
$$

has a smooth kernel. Hence $\dot{A}_z(A^{-1}_{z,B_1} - A^{-1}_{z,B_2})$ is trace class and

$$
\begin{aligned}
\operatorname{Tr}_{L^2}(\dot{A}_z(A^{-1}_{z,B_1} - A^{-1}_{z,B_2})) &= -\operatorname{Tr}_{L^2}(\dot{A}_z\mathcal{K}_z(B_1)B_1P[B_2]^{\perp}P[B_2]^{\perp}\gamma_0A^{-1}_{z,B_2}) \\
&= -\operatorname{Tr}_{W_2^{\perp}}(P[B_2]^{\perp}\gamma_0A^{-1}_{z,B_2}\dot{A}_z\mathcal{K}_z(B_1)B_1P[B_2]^{\perp}) \qquad (3.37)
\end{aligned}
$$

using the fact that $P[B_2]\gamma_0A^{-1}_{z,B_2} = 0$ for the first equality and that $\dot{A}_z\mathcal{K}_z(B_1)B_1P[B_2]^{\perp}$ has a smooth kernel and $P[B_2]^{\perp}\gamma_0A^{-1}_{z,B_2}$ is bounded for the second. Since the operator $B_2^{\perp}\gamma_0A^{-1}_{z,B_2}\dot{A}_z\mathcal{K}_z(B_1)B_1$ is a $\psi$do of order 0, and thus bounded, and $B_1P[B_2]^{\perp}$ is smoothing (*Lemma* 3.6), the trace (3.37) is equal to

$$
\begin{aligned}
&\operatorname{Tr}_{W_1}(B_1\gamma_0A^{-1}_{z,B_2}\frac{d}{dz}(A_z)\mathcal{K}_z(B_1)B_1) \\
=\ &\operatorname{Tr}_{W_1}\left(B_1\gamma_0A^{-1}_{z,B_2}A_{z,B_1}d_z\mathcal{K}_z(B_1)B_1\right) && \text{by (3.34)} \\
=\ &-\operatorname{Tr}_{W_1}\left(S_z(B_1)S_z(B_2)^{-1}B_2\gamma_0d_z\mathcal{K}_z(B_1)B_1\right) && \text{by (3.33)} \\
=\ &-\operatorname{Tr}_{W_1}\left(S_z(B_1)S_z(B_2)^{-1}\frac{d}{dz}\left(S_z(B_2)S_z(B_1)^{-1}B_1\right)\right) && \text{by (3.35)} \\
=\ &\operatorname{Tr}_{H(A_z)}\left(S_z(B_1)^{-1}d_zS_z(B_1) - S_z(B_2)^{-1}d_zS_z(B_2)\right)\ . && (3.38)
\end{aligned}
$$

Here we use the fact that the expression inside the trace in the final equality has a smooth kernel, and so is trace class, in order to swap the order of the operators from the previous line. For $R \neq 0$ in (3.1) it is only the difference in (3.38) that is smoothing. In the product case, though, each term is individually trace class.

On the other hand, the right side of (3.32) is equal to

$$
\begin{aligned}
&\frac{d}{dz}\log\det\nolimits_F\left((S_z(B_1)(\mathcal{E}S_z(B_2))^{-1}\right) \\
=\ &\operatorname{Tr}_{H(A_z)}\left((\mathcal{E}S_z(B_2))S_z(B_1)^{-1}\frac{d}{dz}(S_z(B_1)(\mathcal{E}S_z(B_2))^{-1})\right) \\
=\ &\operatorname{Tr}_{W_1}\left(\mathcal{E}S_z(B_2)S_z(B_1)^{-1}\frac{d}{dz}(S_z(B_1)S_z(B_2)^{-1})\mathcal{E}^{-1}\right)\ ,
\end{aligned}
$$

and this is clearly equal to (3.38) by symmetry of the trace. □

We now have:

**Theorem 3.13.** *Let $A$ be a first-order elliptic operator of Dirac-type and let $E^1 = E^2$. Let $B_1, B_2$ be admissible boundary conditions for $A$ and such that $A_{B_1}, A_{B_2}$ are invertible with common spectral cut $R_\theta$, and such that* (2.13) *holds. Then $A_{B_1}, A_{B_2}$ are $\zeta$-comparable and, relative to a choice of auxiliary operator $\mathcal{E}$, have scattering operator determinant $\det_{\mathcal{E}}(A_{B_1} - \lambda, A_{B_2} - \lambda)$. One has*

$$\det_{\zeta,\theta}(A_{B_1}, A_{B_2}) = \det_{\mathcal{E}}(A_{B_1}, A_{B_2}) \, . \, e^{-\mathrm{LIM}^{\theta}_{\lambda\to\infty} \log\det_{\mathcal{E}}(A_{B_1}-\lambda, A_{B_2}-\lambda)} \; . \tag{3.39}$$

*If $A_{B_1}, A_{B_2}$ are $\zeta$-admissible, then ( in terms of* (3.27)*)*

$$\frac{\det_{\zeta,\theta}(A_{B_1})}{\det_{\zeta,\theta}(A_{B_2})} = \det_F\left(\frac{S(B_1)}{\mathcal{E}S(B_2)}\right) . \, e^{-\mathrm{LIM}^{\theta}_{\lambda\to\infty} \log\det_F((\mathcal{E}S_\lambda(B_2))^{-1}S_\lambda(B_1))} \; . \tag{3.40}$$

*Proof.* From Proposition 3.1 we have that $B_1, B_2$ are well-posed and admissible for $A_\lambda := A - \lambda$, while (3.32) becomes

$$\mathrm{Tr}\,((A_{B_1} - \lambda)^{-1} - (A_{B_2} - \lambda)^{-1}) = -\frac{\partial}{\partial\lambda}\log\det_F\left(\frac{S_\lambda(B_1)}{\mathcal{E}S_\lambda(B_2)}\right) \; , \tag{3.41}$$

Hence, with the stated assumptions, $A_{B_1}, A_{B_2}$ are $\zeta$-comparable and (3.39) is immediate from Theorem 2.5 and (3.27). Finally, (3.40) follows from Lemma 2.3 and (2.30). □

That the right-sides of (3.39), (3.40) are independent of the choice of $\mathcal{E}$ is clear from (3.28) and (2.44). More precise knowledge of the dependence of the regularized limit on the operators $A$ and $B_i$ is obtained as follows.

**Proposition 3.14.** *With the conditions of Proposition 3.12,*

$$\frac{d}{dz}\log\det_{\zeta,\theta}(A_{z,B_1}, A_{z,B_2}) = \frac{d}{dz}\log\det_{\mathcal{E}}(A_{z,B_1}, A_{z,B_2}) \; , \tag{3.42}$$

*and hence is independent of $\theta$. Moreover, with $\zeta_{z,rel}(0) := \zeta_\theta(A_{z,B_1}, A_{z,B_2}, 0)$*

$$\frac{d}{dz}\zeta_{z,rel}(0) = 0 \; . \tag{3.43}$$

*The regularized limit term in Theorem 3.13 is independent of the operator $A$, and depends only on the pseudodifferential boundary conditions $B_1, B_2, P(A)$.*

*Proof.* $(A_{z,B_1} - \lambda)^{-1} - (A_{z,B_2} - \lambda)^{-1}$ has a smooth kernel, and hence the operator $J(\lambda) = \dot{A}_z((A_{z,B_1} - \lambda)^{-1} - (A_{z,B_2} - \lambda)^{-1})$ is trace class. A well known argument [8] gives $(d/dz)(\mathrm{Tr}\,((A_{z,B_1}-\lambda)^{-1}-(A_{z,B_2}-\lambda)^{-1})) = -\partial_\lambda\Phi(\lambda)$ , where $\Phi(\lambda) = \mathrm{Tr}\,(J(\lambda))$. Hence

$$-\frac{d}{dz}\zeta_\theta(A_{z,B_1}, A_{z,B_2}, s) = \frac{i}{2\pi}\int_C \lambda^{-s}\partial_\lambda\Phi(\lambda)\; d\lambda \; , \tag{3.44}$$

and so from Proposition 2.9, $-\frac{d}{dz}\zeta'_\theta(A_{z,B_1}, A_{z,B_2}, 0) = \mathrm{Tr}\,(J(0)) - \mathrm{LIM}^{(\theta)}_{\lambda\to\infty}\mathrm{Tr}\,(J(\lambda)$. By (3.32), then, (3.42) is equivalent to $\mathrm{LIM}^{(\theta)}_{\lambda\to\infty}\mathrm{Tr}\,(J(\lambda)) = 0$. To see that, for $\mathrm{Re}(s) > 1 - \alpha_0$ we can integrate by parts in (3.44) to obtain

$$-\frac{d}{dz}\zeta_\theta(A_{z,B_1}, A_{z,B_2}, s) = s\mathrm{Tr}\,(\dot{A}_z(A^{-s-1}_{z,B_1} - A^{-s-1}_{z,B_2})) \; . \tag{3.45}$$

But $A^{-s-1}_{z,B_1} - A^{-s-1}_{z,B_2}$ has a smooth kernel for $\mathrm{Re}(s) > -1$ and hence (3.45) holds in that larger half-plane. Setting $s = 0$ in (3.45) therefore proves (3.43), while differentiating and setting $s = 0$ we obtain

$$-\frac{d}{dz}\zeta'_\theta(A_{z,B_1}, A_{z,B_2}, 0) = \mathrm{Tr}\,(\dot{A}_z(A^{-1}_{z,B_1} - A^{-1}_{z,B_2}))\ , \tag{3.46}$$

which is equation (3.42).

For the final statement, let $A_r, -\epsilon < r < \epsilon$ be a smooth path of Dirac-type operators, as in Proposition 3.12, with $A_0 = A$. The variation near $\partial X$ is at most order 0 and so from Proposition 3.1 $B_1, B_2$ are admissible for each $A_r$. For small enough $\epsilon$ we can apply (3.42) and comparing with (3.39) we reach the conclusion. □

If $A_r,\ 0 \le r \le t$, is a smooth 1-parameter family satisfying (3.30) with $A_{r,B_i}$ invertible, then the final statement of Proposition 3.14 can equivalently be expressed by the integrated version of (3.42):

$$\frac{\det_{\zeta,\theta}(A_{t,B_1}, A_{t,B_2})}{\det_{\zeta,\theta}(A_{0,B_1}, A_{0,B_2})} = \det{}_F\left(\frac{S_t(B_1)}{\mathcal{E}S_t(B_2)}\right)/\det{}_F\left(\frac{S_0(B_1)}{\mathcal{E}S_0(B_2)}\right)\ .$$

Next, we compute the variation of the $\zeta$-determinant with respect to the boundary condition. The following formula gives a general direct variational formula[5].

**Proposition 3.15.** *Let $\{B_r \mid -\epsilon < r < \epsilon\}$ be a smooth 1-parameter family of* $\psi$dos *on $L^2(Y, E^1_{|Y})$ such that $B_r - P(A)$ has a smooth kernel and such that $A_{B_r}$ is invertible for each $r$ and $\zeta$-admissible. Then setting $S_\lambda(B_r) = S_{(A-\lambda)}(B_r)$, one has*

$$\frac{d}{dr}\log\det{}_{\zeta,\theta}(A_{B_r}) = \mathrm{Tr}\left(S(B_r)^{-1}\frac{d}{dr}S(B_r)\right) - \mathrm{LIM}^\theta_{\lambda\to\infty}\mathrm{Tr}\left(S_\lambda(B_r)^{-1}\frac{d}{dr}S_\lambda(B_r)\right)\ . \tag{3.47}$$

*Proof.* Each $B_r$ is an admissible boundary condition for the Dirac-type operator $A_\lambda := A - \lambda$ and $\dot{B}_r = d/dr(B_r)$ is a smoothing operator on $L^2(Y, E^1_{|Y})$. We have

$$\frac{d}{dr}\frac{\partial}{\partial\lambda}\log\det{}_F\left(\frac{S_\lambda(B_r)}{\mathcal{E}S_\lambda(B_0)}\right) = \frac{\partial}{\partial\lambda}\mathrm{Tr}_{\,H(A_\lambda)}(S_\lambda(B_r)^{-1}\frac{d}{dr}S_\lambda(B_r))\ . \tag{3.48}$$

Hence from (3.41)

$$-\frac{d}{dr}\zeta_\theta(A_{B_r}, s) = -\frac{d}{dr}\zeta_\theta(A_{B_r}, A_{B_0}, s) = -\frac{i}{2\pi}\int_C \lambda^{-s}\frac{\partial}{\partial\lambda}\mathrm{Tr}\,(S_\lambda(B_r)^{-1}\frac{d}{dr}S_\lambda(B_r))\,d\lambda\ ,$$

and so the result follows from (2.48). □

[5] The usual approach to computing the boundary variation is to try to 'gauge transform' the variation into an equivalent operator variation, see [7, 28, 20] and also §4.1.

3.4. **Local Coordinates and an Odd-Dimensional Example.** The identity (3.40) can be given a more familiar form if we work in local coordinates on $Gr_\infty(D)$.

• **The relative zeta determinant in Stiefel Coordinates**

To be concrete, let $X$ be a compact Riemannian spin manifold and consider a compatible Dirac operator $A = D : C^\infty(X, E^1) \to C^\infty(X, E^2)$ acting between Clifford bundles in the product case ($R = 0$). First, observe that to each basepoint $\Pi \in Gr_\infty^{(0)}(D)$ there is a dense open subset of $Gr_\infty^{(0)}(D)$. Setting

$$E = \mathrm{ran}\,(\Pi) \ , \qquad W = \mathrm{ran}\,(P) \ , \qquad W_i = \mathrm{ran}\,(P_i) \ , \tag{3.49}$$

it is defined by

$$U_E = \{P \in Gr_\infty^{(0)}(D) \mid (\Pi, P) := P \circ \Pi : E \to W \text{ invertible}\} \ . \tag{3.50}$$

Equivalently,

$$P \in U_E \Longleftrightarrow \mathrm{ran}\,(P) = \mathrm{graph}(T : E \to E^\perp) \qquad T \in \mathrm{Hom}_\infty(E, E^\perp) \ , \tag{3.51}$$

where $\mathrm{Hom}_k(E, E^\perp) = \{\Pi^\perp \mathsf{Z} \Pi : E \to E^\perp \mid \mathsf{Z} \in \Psi_k(H_Y)\}$ and $\Psi_k(H_Y)$ is the space of $\psi$dos on $H_Y = L^2(Y, E^1_{|Y})$ of order $-k \in \mathbb{R}_+ \cup \{\infty\}$. Equivalently,

$$P \in U_E \Longleftrightarrow P = P_T = \begin{pmatrix} Q_T^{-1} & Q_T^{-1}T^* \\ TQ_T^{-1} & TQ_T^{-1}T^* \end{pmatrix}, \qquad Q_T := I + T^*T \ , \tag{3.52}$$

$T \in \mathrm{Hom}_\infty(E, E^\perp)$. In this way a atlas for $Gr_\infty^{(0)}(D)$ can be constructed with respect to a countable set of basepoint spectral projections in a similar way to [22].

It is always possible to arrange for $P(D), P_1, P_2$ to lie in a single coordinate patch $U_E \subset Gr_\infty^{(0)}(D)$, so that,

$$H(D) = \mathrm{graph}(K : E \to E^\perp) \ , \qquad W_i = \mathrm{graph}(T_i : E \to E^\perp) \ , \tag{3.53}$$

where $K, T_i \in \mathrm{Hom}_\infty(E, E^\perp)$. Since the operators $S(P_i) = P_iP(D)$ are invertible, one such choice is $E = H(D)$, in which case $K = 0$. We may further assume, by perturbing $\theta$ slightly if necessary, that the global boundary problem $D_\Pi$ has no eigenvalue along $R_\theta$, and hence that $\Pi \circ P(D-\lambda) : H(D-\lambda) \to \mathrm{ran}\,(\Pi)$ is invertible. This means that

$$H(D-\lambda) = \mathrm{graph}(K_\lambda : E \to E^\perp) \ , \qquad P(D-\lambda) = P_{K_\lambda} \ , \tag{3.54}$$

for some unique $K_\lambda \in \mathrm{Hom}_1(E, E^\perp)$ (the space of restrictions of $\psi$dos of order $-1$). Recall that $P(D-\lambda)$ is an element of $Gr_1(D)$, though not of $Gr_\infty(D)$ if $\lambda \neq 0$.

We then have:

**Proposition 3.16.**

$$\det{}_F\left(\frac{S_\lambda(P_1)}{\mathcal{E}S_\lambda(P_2)}\right) = \det{}_F\left(\frac{I + T_1^*K_\lambda}{I + T_2^*K_\lambda}\right) . \det{}_F(\Phi(\mathcal{E}, T_1, T_2)) \ , \tag{3.55}$$

*where the Fredholm determinants are taken on $E$. The operator $\Phi(\mathcal{E}, T_1, T_2) : E \to E$ is invertible and independent of $K_\lambda$. If $(P_2, P_1)$ is invertible, then*

$$\det_F\left(\frac{S_\lambda(P_1)}{P_1 S_\lambda(P_2)}\right) = \det_F\left(\frac{I + T_1^* K_\lambda}{I + T_2^* K_\lambda}\right) . \det_F(Q_2(I + T_1^* T_2)^{-1}) \ . \tag{3.56}$$

*Proof.* Let $P, \tilde{P} \in Gr_1(D)$ with $\mathrm{ran}\,(P) = \mathrm{graph}(T)$, $\mathrm{ran}\,(\tilde{P}) = \mathrm{graph}(\tilde{T})$, where $T, \tilde{T} \in \mathrm{Hom}_1(E, E^\perp)$. Then any linear operator $R : \mathrm{ran}\,(P) \to \mathrm{ran}\,(\tilde{P})$ acts by

$$(\xi, T\xi) \longmapsto (\Phi(R)\xi, \tilde{T}\Phi(R)\xi) \ ,$$

for some $\Phi(R) \in \mathrm{End}(E)$. $\Phi$ respects operator composition: if $\tilde{R} : \mathrm{ran}\,(\tilde{P}) \to \mathrm{ran}\,(P)$

$$\Phi(\tilde{R}R) = \Phi(\tilde{R})\Phi(R) \ . \tag{3.57}$$

Moreover, if $\tilde{R}R : \mathrm{ran}\,(P) \to \mathrm{ran}\,(P)$ is determinant class, then so is $\Phi(\tilde{R}R)$ and

$$\det_F(\tilde{R}R) = \det_F(\Phi(\tilde{R}R)) \ , \tag{3.58}$$

where the left-side is taken on $\mathrm{ran}\,(P)$ and the right-side on $E$.

In particular, the auxiliary operator $\mathcal{E}$ acts via $\Phi(\mathcal{E}) \in \mathrm{End}(E)$. Similarly, $(P_2, P_1)$ acts via $\Phi(P_2, P_1) \in \mathrm{End}(E)$ and with $Q_i := Q_{T_i}$ one has using (3.52)

$$(P_2, P_1) := P_1 \circ P_2 \begin{pmatrix} \xi \\ T_2\xi \end{pmatrix} = \begin{pmatrix} Q_1^{-1}(I + T_1^* T_2)\xi \\ T_1 Q_1^{-1}(I + T_1^* T_2)\xi \end{pmatrix}$$

and so

$$\Phi(P_2, P_1) = Q_1^{-1}(I + T_1^* T_2) \ . \tag{3.59}$$

Hence $(P_2, P_1)$ is invertible when $-1 \notin \mathrm{sp}(T_1^* T_2)$. On the other hand, since $\mathcal{E}$ is invertible so is $\Phi(\mathcal{E})$, and because $\mathcal{E} - (P_2, P_1)$ is trace-class then $\Phi(\mathcal{E}) - Q_i^{-1}(I + T_1^* T_2)$ is also trace-class. Hence $\Phi(\mathcal{E})$ is of determinant class and $\det_F(\Phi(\mathcal{E})) \neq 0$. It is easy to compute that

$$\Phi(S_\lambda(P_1)) = Q_1^{-1}(I + T_1^* K_\lambda) \ , \qquad \Phi(\mathcal{E} S_\lambda(P_2)) = \Phi(\mathcal{E}) Q_2^{-1}(I + T_2^* K_\lambda) \ .$$

From (3.57), (3.58), and the symmetry and multiplicativity of $\det_F$, then by setting $\Phi(\mathcal{E}, T_1, T_2) = Q_2 \Phi(\mathcal{E})^{-1} Q_1^{-1}$ we reach the conclusion.

Alternatively, since $(\mathcal{E} S_\lambda(P_2))^{-1} S_\lambda(P_1)) = S_\lambda(P_2)^{-1} \mathcal{E}^{-1} S_\lambda(P_1)$, the computation can be carried through by observing that for any invertible operator $R$ as above, one has (relative to graph coordinates)

$$P R^{-1} \tilde{P} = \begin{pmatrix} \Phi(R)^{-1} & \Phi(R)^{-1}\tilde{T}^* \\ T\Phi(R)^{-1} & T\Phi(R)^{-1}\tilde{T}^* \end{pmatrix} \ . \tag{3.60}$$

□

We can now restate Theorem 3.13 as follows:

**Theorem 3.17.** *Let $D$ be a first-order Dirac-type operator in the product case and let $E^1 = E^2$. Let $P_1, P_2 \in Gr_\infty(D)$ such that $D_{P_1}, D_{P_2}$ are invertible with common spectral cut $R_\theta$, and such that* (2.13) *holds. Then $D_{P_1}, D_{P_2}$ are $\zeta$-comparable and in local Stiefel (graph) coordinates, as above, one has*

$$\frac{\det_{\zeta,\theta}(D_{P_1})}{\det_{\zeta,\theta}(D_{P_2})} = \frac{\det_F(I + T_1^* K)}{\det_F(I + T_2^* K)} \cdot \exp\left[-\mathrm{LIM}^\theta_{\lambda\to\infty} \log \det_F \left(\frac{I + T_1^* K_\lambda}{I + T_2^* K_\lambda}\right)\right] . \tag{3.61}$$

*Proof.* This is immediate from (3.40), (2.44), (3.55). Because $P(D) \in Gr_\infty(D)$ we have replaced the determinant of the quotient by the quotient of the determinants in the first term on the right-side of (3.61). □

**Remark 3.18.** *More generally, Stiefel coordinates on $Gr^{(0)}_\infty(D)$ refer to an operator $[M\ \ N] \in \mathrm{Hom}(E \oplus E^\perp, E)$, where $M$ is Fredholm with $\mathrm{ind}\,(M) = 0$, and $N \in \mathrm{Hom}_\infty(E^\perp, E)$. This defines a point in the principal Stiefel frame bundle $ST_E \longrightarrow Gr^{(0)}_\infty(D)$ (based at $E$), with bundle projection map*

$$[M\ N] \longmapsto P := \begin{pmatrix} M^*\mathcal{M}^{-1}M & M^*\mathcal{M}^{-1}N \\ N^*\mathcal{M}^{-1}M & N^*\mathcal{M}^{-1}N \end{pmatrix} , \tag{3.62}$$

*where $\mathcal{M} = MM^* + NN^*$. In particular, graph coordinates correspond to the canonical section $P_T \mapsto [I\ T^*]$ of $ST_E$ over $U_E$. Stiefel coordinates $[M_i\ N_i]$ for $P_i$ modify* (3.61) *by replacing $I + T_i^* K_\lambda$ by $M_i + N_i K_\lambda : E \to E$.*

• **Example: odd-dimensions revisited**

To illustrate these formulae, we explain how they work for a Dirac operator with $X$ odd-dimensional. In this case (3.1) takes the form

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\partial_u + \begin{pmatrix} 0 & \mathsf{D}_Y^- \\ \mathsf{D}_Y^+ & 0 \end{pmatrix}\right) \tag{3.63}$$

with respect to the decomposition $H_Y = F^+ \oplus F^-$ into chiral spinor fields, where $\mathsf{D}_Y^+$ is the chiral Dirac operator, which, for brevity, we shall assume invertible. The projection $P_{F^+}$ onto $F^+$ is not an element of $Gr_1(D)$. It does, however, define a true (local) elliptic boundary condition and is related to $\Pi_\geq$ in the following precise way.

The involution defining the grading of $H_Y$ into positive and negative energy (the APS condition) is the operator

$$\mathcal{D}_Y^{-1}|\mathcal{D}_Y| = \begin{pmatrix} 0 & (\mathsf{D}_Y^+)^{-1}(\mathsf{D}_Y^+\mathsf{D}_Y^-)^{1/2} \\ (\mathsf{D}_Y^+\mathsf{D}_Y^-)^{-1/2}\mathsf{D}_Y^+ & 0 \end{pmatrix} .$$

Hence, defining $g_+$ to be the unitary isomorphism $(\mathsf{D}_Y^+\mathsf{D}_Y^-)^{-1/2}\mathsf{D}_Y^+ : F^+ \to F^-$, we have

$$\Pi_\geq = \frac{1}{2}(I + \mathcal{D}_Y^{-1}|\mathcal{D}_Y|) = \frac{1}{2}\begin{pmatrix} I & g_+^{-1} \\ g_+ & I \end{pmatrix} .$$

The global boundary problem $D_{\Pi_\geq}$ is self-adjoint and, more generally, a boundary condition $P \in Gr_\infty(D)$ such that $D_P$ is self-adjoint is characterized by having range

equal to the graph of an $L^2$-unitary isomorphism $T : F^+ \to F^-$ such that $T - g_+$ has a smooth kernel [24]. Thus each self-adjoint boundary condition $P = P_T$ defines a point $\det(T)$ of the determinant line

$$\mathrm{Det}\,(T) = \mathrm{Det}\,(g_+) \cong \mathrm{Det}\,((\mathsf{D}_Y^+\mathsf{D}_Y^-)^{-1/2}) \otimes \mathrm{Det}\,(\mathsf{D}_Y^+) \cong \mathrm{Det}\,(\mathsf{D}_Y^+) \tag{3.64}$$

of the boundary chiral Dirac operator $\mathsf{D}_Y^+$. The first isomorphism in (3.64) is a general functorial property of determinant lines under composition of Fredholm operators [26], while the second is defined through the $\zeta$-determinant

$$\det_\zeta : \mathrm{Det}\,(\mathsf{D}_Y^+\mathsf{D}_Y^-) \longrightarrow \mathbb{C}\ .$$

That this map is a *linear* isomorphism is a consequence of Proposition 2.19. A consequence of (3.64) and Theorem 2.16 is that the $\eta$-invariant defines a canonical linear isomorphism

$$e^{2\pi i\widetilde{\eta}(D)} : \mathrm{Det}\,(\mathsf{D}_Y^+) \longrightarrow \mathbb{C}\ , \tag{3.65}$$

via (3.64) and the assignment $T \longmapsto e^{2\pi i\widetilde{\eta}(D_{P_T})}$. That is:

**Proposition 3.19.** *Absolutely—without a choice of boundary condition—the exponentiated eta-invariant is a canonical element of the dual determinant line of the boundary Dirac operator*[6]

$$e^{2\pi i\widetilde{\eta}(D)} \in \mathrm{Det}\,(D_Y^+)^*\ . \tag{3.66}$$

To see this, first observe since $T$ is unitary that (3.52) becomes

$$P_T = \frac{1}{2}\begin{pmatrix} I & T^{-1} \\ T & I \end{pmatrix}\ .$$

In particular, this holds for $P(D)$ for some unique unitary $K : F^+ \to F^-$. It does not quite hold for $D - \lambda$ since the operator is not of product type, but it is still true that $H(D - \lambda)$ is the graph of a $\psi$do operator $K_\lambda : F^+ \to F^-$ of order 0, though not that $K_\lambda$ is an isometry or that $K_\lambda - g_+$ is smoothing.

Consider two 'self-adjoint' boundary conditions $P_1 = P_{T_1}, P_2 = P_{T_2} \in Gr_\infty(D)$. The spectrum of the operators $D_{P_i}$ is real and unbounded and, as in § 2.2, we denote the two choices for $\theta$ by $\pm$.

**Theorem 3.20.** [28] *For self-adjoint global boundary problems $D_{P_1}, D_{P_2}$ for the Dirac operator over an odd-dimensional spin manifold*

$$\frac{\det_{\zeta,\pm}(D_{P_1})}{\det_{\zeta,\pm}(D_{P_2})} = \frac{\det_F\left(\frac{1}{2}(I + (T_1^{-1}K)^{\mp 1})\right)}{\det_F\left(\frac{1}{2}(I + (T_2^{-1}K)^{\mp 1})\right)}\ . \tag{3.67}$$

*Equivalently, if $P = P_T$*

$$\det_{\zeta,\pm}(D_P) = \det_{\zeta,\pm}(D_{P(D)})\,.\,\det_F\left[\frac{1}{2}(I + (T^{-1}K)^{\mp 1})\right]\ . \tag{3.68}$$

[6]The was first pointed out by Segal [33].

*Proof.* The equality (3.61) becomes

$$\begin{aligned} \frac{\det_{\zeta,\pm}(D_{P_1})}{\det_{\zeta,\pm}(D_{P_2})} &= \det{}_F\left(\frac{I+T_1^{-1}K}{I+T_2^{-1}K}\right) . \exp\left[-\mathrm{lim}^{\pm}_{\lambda\to\infty}\log\det{}_F\left(\frac{I+T_1^{-1}K_\lambda}{I+T_2^{-1}K_\lambda}\right)\right] \\ &= \frac{\det_F\left(\frac{1}{2}(I+T_1^{-1}K)\right)}{\det_F\left(\frac{1}{2}(I+T_2^{-1}K)\right)} . \exp\left[-\mathrm{lim}^{\pm}_{\lambda\to\infty}\log\det{}_F\left(\frac{I+T_1^{-1}K_\lambda}{I+T_2^{-1}K_\lambda}\right)\right] . \end{aligned} \tag{3.69}$$

The only extra subtlety introduced by $P_{F^+}\notin Gr_\infty(D)$ is that it is only the quotient of operators $(I+T_1^{-1}K)/(I+T_2^{-1}K)$ which has a Fredholm determinant. But since $T_i^{-1}K$ is of determinant class then so is $(1/2)(I+T_1^{-1}K)$. From [34] we have that $D_{P_1}, D_{P_2}$ are strongly $\zeta$-comparable and hence LIM becomes the usual lim (§ 2.4).

Finally, either directly, using $g_\lambda^{-1}(D-\lambda)_{|U}g_\lambda = D_{|U}$ with $g_\lambda = e^{-iu\lambda}\oplus e^{iu\lambda}$ in the collar $U$, or using the symmetry argument of [28], one has

$$K_\lambda \to 0 \ \text{ as } \ \lambda\to\infty \ \text{ on } \ R_- , \qquad K_\lambda^{-1}\to 0 \ \text{ as } \ \lambda\to\infty \ \text{ on } \ R_+ . \tag{3.70}$$

The conclusion then follows from (3.69). □

From (3.67), switching the spectral cut conjugates the relative zeta determinant. This corresponds to the equivalent descriptions of $\mathrm{ran}\,(P_T)$ as $\mathrm{graph}(T:F^+\to F^-)$ or $\mathrm{graph}(T^{-1}:F^-\to F^+)$[7]. More generally, allowing $D_{P_i}$ to be non-invertible, this disparity derives from the relative eta-invariant:

**Theorem 3.21.**

$$\widetilde{\eta}(D_{P_1}) - \widetilde{\eta}(D_{P_2}) = \frac{1}{2\pi i}\log\det{}_F(T_2^{-1}T_1) \qquad \mathrm{mod}\ (\mathbf{Z}) . \tag{3.71}$$

*Proof.* From [34], $D_{P_1}, D_{P_2}$ are strongly $\zeta$-comparable and $\zeta(D_{P_1}^2, D_{P_2}^2, 0) = 0$. Hence from (2.84), (3.55) we have mod $(2\pi i\mathbf{Z})$

$$\begin{aligned} 2\pi i\widetilde{\eta}(D_{P_1},D_{P_2}) &= \mathrm{lim}_{\alpha\to+\infty}\left(\log\det{}_F\mathcal{S}_{-i\alpha} - \log\det{}_F\mathcal{S}_{i\alpha}\right) \\ &= \mathrm{lim}_{\alpha\to+\infty}\left[\log\det{}_F\left(\frac{I+T_1^{-1}K_{-i\alpha}}{I+T_2^{-1}K_{-i\alpha}}\right) - \log\det{}_F\left(\frac{I+T_1^{-1}K_{i\alpha}}{I+T_2^{-1}K_{i\alpha}}\right)\right] \\ &= \log\det{}_F\left(T_2^{-1}T_1\right) \qquad \mathrm{mod}\ (2\pi i\mathbf{Z}) \end{aligned}$$

where the final equality follows from (3.70). Since $D_{P_1}, D_{P_2}$ are $\zeta$-admissible, then (2.73) completes the proof. □

The identity (3.71) is deduced in [14] from (3.67). Notice that (3.71) is independent of $K:F^+\to F^-$. More precisely, from (3.24), equation (3.71) is the assertion that (3.65) is linear.

**Remark 3.22.** *(1) For a smooth family of $\zeta$-admissible operators* $\det_\zeta$ *defines a section of the dual determinant line bundle, that is, an element of Fock space. For a family of of self-adjoint boundary problems $D_P$,* $\det_\zeta$ *defines a an element of the Fock*

[7] Like the $\zeta$-determinant, the 'canonical determinant' of [24, 28] is therefore not quite canonical. The only completely canonical boundary determinant is the quotient (3.29), which, like the relative $\zeta$-determinant, has no 'parity anomaly'.

*space associated to $X$ (this is* (3.67)*), while the exponentiated $\eta$-invariant defines an element of the boundary Fock space (this is* (3.66),(3.71)*).*
*(2) The extension to the case where $\mathcal{D}_Y^+$ is non-invertible is easily done by augmenting $\mathcal{D}_Y^+$ by a unitary isomorphism $\sigma : \mathrm{Ker}(\mathcal{D}_Y^+) \to \mathrm{Ker}(\mathcal{D}_Y^-)$. In particular, Thm(2.21) of* [20]*, Thm(3.1) of* [16]*, are special cases of* (3.71)*.*

It is worth pointing out that, since $\mathrm{ran}\,(P_T^{\perp}) = \mathrm{graph}(-T^{-1} : F^- \to F^+)$, if $M = X \cup X'$ is a closed manifold with Dirac operator $A$ with $A_{|X} = D$ and $A_{|X'} := D'$, then an easy corollary of (3.71) is the

*Weak Splitting Theorem : $\eta(D_{P_T}) + \eta(D'_{P_T^{\perp}})$ is constant as $P_T$ varies.*

The hard splitting Theorem [5, 14, 20, 34] asserts this constant is precisely $\eta(A)$.

Another way of viewing the conjugation of the relative zeta determinant on taking the conjugate spectral cut is through the following formula for the relative Laplacian:

**Proposition 3.23.**

$$\frac{\det_{\zeta,\pi}(D^2_{P_1})}{\det_{\zeta,\pi}(D^2_{P_2})} = \left|\frac{\det_{\zeta,\pm}(D_{P_1})}{\det_{\zeta,\pm}(D_{P_2})}\right|^2 = \left|\frac{\det_F\left(\frac{1}{2}(I + T_1^{-1}K\right)}{\det_F\left(\frac{1}{2}(I + T_2^{-1}K)\right)}\right|^2 .$$

*Proof.* Immediate from (2.79), (3.55), (3.71). ☐

This formula is a special case of Theorem A to which we turn next. (See also Remark 4.3(2)).

## 4. An Application to the Laplacian on a Manifold with Boundary

Let $X$ be an $n$-dimensional $C^\infty$ compact Riemannian manifold with boundary $Y$ and let $D : C^\infty(X, E^1) \longrightarrow C^\infty(X, E^2)$ be a Dirac-type operator with product case geometry, so that

$$D_{|U} = \sigma\left(\frac{\partial}{\partial u} + \mathcal{D}_Y\right), \tag{4.1}$$

in a collar $U = [0, 1) \times Y$ of the boundary, with notation as in (3.1).

For each well-posed boundary condition $B$ for $D$ the associated Dirac Laplacian

$$\Delta_B = D_B^* D_B = D^* D : \mathrm{dom}\,(\Delta_B) \to L^2(X, E^1)$$

with domain

$$\mathrm{dom}\,(\Delta_B) = \{\psi \in H^2(X, E^1) \mid B\gamma_0\psi = 0, B^\star\gamma_0 D\psi = 0\}$$

is a closed self-adjoint and positive operator on $L^2(X, E^1)$ with discrete non-negative real spectrum.

The following result of Grubb allows us to define the zeta determinant of $\Delta_B$.

**Proposition 4.1.** [10, 11] *If $B$ is an admissible well-posed boundary condition for $D$, then $\Delta_B$ is $\zeta$-admissible with spectral cut $R_\pi$.*

More precisely, Grubb proves that there is a resolvent trace expansion for $m > n/2$ as $\lambda \to \infty$ in closed subsectors of $\mathbb{C}\backslash\overline{\mathbb{R}}_+$

(4.2)
$$\mathrm{Tr}\,\left(\partial_\lambda^m(\Delta_B - \lambda)^{-1}\right) \sim \sum_{j=-n}^{-1} a_j(-\lambda)^{-j/2-m-1} + \sum_{j=0}^{\infty}(a_{j,k}\log(-\lambda) + c_j)(-\lambda)^{-j/2-m-1}$$

and hence that the $\zeta$-function $\zeta(\Delta_B, s)$ defined by the standard trace $\mathrm{Tr}\,(\Delta_B^{-s})$ for $\mathrm{Re}(s) > n/2$ extends meromorphically to all of $\mathbb{C}$ with the singularity structure

$$\Gamma(s)\zeta(\Delta_B, s) \sim \sum_{j=-n}^{-1} \frac{\tilde{a}_j}{s + k/2} + \frac{\dim\ker(\Delta_B)}{s} + \sum_{j=0}^{\infty}\left(\frac{\tilde{a}_j}{(s+k/2)^2} + \frac{\tilde{c}_j}{s+k/2}\right)\ , \tag{4.3}$$

where $\theta = \pi$ and the coefficients in (4.3) differ from those in (4.2) by universal constants. If $B - P(A)$ is a $\psi$do of order $\leq -n$ then the coefficient $\tilde{a}_0$ vanishes and so $\zeta(\Delta_B, s)$ is then regular at $s = 0$ and

$$\mathrm{det}_\zeta\Delta_B = \exp(-\zeta^{'}(\Delta_B, 0))$$

is well-defined. In particular, $\det_\zeta \Delta_P$ exists for all $P \in Gr_\infty(D)$.

On the other hand, setting $S(P) := S_D(P)$, we have from Proposition 3.5 that the boundary 'Laplacian'

$$S(P)^*S(P) = P(D)^* \cdot P \cdot P(D) : H(D) \longrightarrow H(D)$$

is of determinant class for all $P \in Gr_\infty(D)$.

The main purpose of this section is to prove the following Theorem.

**Theorem 4.2.** *Let $B_1, B_2$ be admissible well-posed boundary conditions for a Dirac-type operator $D : C^\infty(X, E^1) \longrightarrow C^\infty(X, E^2)$. Then, with $P_i = P[B_i]$, one has*

$$\frac{\det_\zeta(\Delta_{B_1})}{\det_\zeta(\Delta_{B_2})} = \frac{\det_F\,(S(P_1)^*S(P_1))}{\det_F\,(S(P_2)^*S(P_2))}\ . \tag{4.4}$$

*Or, from* (2.100),

$$\det_\zeta(\Delta_{B_1}, \Delta_{B_2}) = \det_\zeta(S(P_1)^*S(P_1)\ ,\ S(P_2)^*S(P_2))\ . \tag{4.5}$$

*Equivalently, since $S(P(D)) = Id$,*

$$\det_\zeta(\Delta_B) = \det_\zeta(\Delta_{P(D)}).\det_F\,(S(P[B])^*S(P[B]))\ . \tag{4.6}$$

**Remark 4.3.** *(1) Because of Lemma 3.6 and* (3.7) *it is sufficient to assume that $B_i = P_i \in Gr_\infty(D)$, and from here on that is what we shall do.*
*(2) In Stiefel graph coordinates* (4.4) *has the form*

$$\frac{\det_\zeta(\Delta_{P_1})}{\det_\zeta(\Delta_{P_2})} = \frac{\det_F\left(Q_1^{-1}\,(I + T_1^*K)\,(I + K^*T_1)\right)}{\det_F\left(Q_2^{-1}\,(I + T_2^*K)\,(I + K^*T_2)\right)} = e^{\log\det_F(Q_1^{-1}Q_2)}\left|\frac{\det_F\,(I + T_1^*K)}{\det_F\,(I + T_2^*K)}\right|^2.$$

*(3) The simple form of* (4.4) *depends on the homogeneous structure of $Gr_\infty(D)$, it does not persist to more general classes of well-posed boundary conditions.*

*(4) We may replace $P(D)$ by $P(D)_{\rm ort}$ (cf. (3.4)) in Theorem 4.2). This follows from the invertibility of $P(D)(P(D)P(D)^* + (I - P(D)^*)(I - P(D)))^{-1}P(D)^*P(D)$ on $H(D)$, which is therefore not detected in the quotient on the right-side of (4.4).*
*(5) Implicit in Theorem 4.2 is the invertibility of $D_{B_i}$. We assume invertibility when obviously required without further mention. The identity (4.6) is globally defined.*
*(6) The identifications hold for $P_1 - P_2$ differing just by a $\psi$do of order $< -n$.*

4.1. **Proof of Theorem 4.2.** To identify the scattering operator we use a canonical identification of the solution space of $\Delta_P$ with that of an associated first-order elliptic system.

- **An equivalent first-order elliptic system**

We analyze $\Delta_P = D_P^* D_P$ through the first-order elliptic operator acting on sections of $E^1 \oplus E^2$

$$\widehat{\Delta} = \begin{pmatrix} 0 & D^* \\ D & -I \end{pmatrix} : H^1(X; E^1 \oplus E^2) \to L^2(X; E^1 \oplus E^2) \ .$$

$$\widehat{\Delta}(s_1, s_2) = (D^* s_2, D s_1 - s_2) \ .$$

From (3.1) and (3.13) we find that $\widehat{\Delta}$ is of Dirac-type with

$$\widehat{\Delta}_{|U} = \widehat{\sigma}\left(\frac{\partial}{\partial u} + \widehat{\mathcal{A}}_Y + \widehat{R}\right),$$

where

$$\widehat{\sigma} = \begin{pmatrix} 0 & -\sigma^{-1} \\ \sigma & 0 \end{pmatrix}, \qquad \widehat{\mathcal{A}}_Y = \begin{pmatrix} \mathcal{D}_Y & 0 \\ 0 & -\sigma \mathcal{D}_Y \sigma^{-1} \end{pmatrix}, \qquad \widehat{R} = \begin{pmatrix} 0 & -\sigma^{-1} \\ 0 & 0 \end{pmatrix}, \tag{4.7}$$

satisfying the relations

$$\widehat{\sigma}^2 = -I \ , \qquad \widehat{\sigma}^* = -\widehat{\sigma} \ , \qquad \widehat{\sigma}\widehat{\mathcal{A}}_Y + \widehat{\mathcal{A}}_Y\widehat{\sigma} = 0 \ , \qquad \widehat{\sigma}\widehat{R} + \widehat{R}\widehat{\sigma} = -I \ . \tag{4.8}$$

Green's Theorem for the formally self-adjoint operator $\widehat{\Delta}$ now states that

$$< \widehat{\Delta} s_1, s_2 > - < s_1, \widehat{\Delta} s_2 > = < -\widehat{\sigma}\gamma_0 s_1, \gamma_0 s_2 > , \tag{4.9}$$

where here $\gamma_0(\psi, \phi) = (\gamma_0\psi, \gamma_0\phi)$.

Setting $A = \widehat{\Delta}$ in our discussion in §4, we have a Poisson operator for $\widehat{\Delta}$

$$\widehat{\mathcal{K}} := \mathcal{K}_{\widehat{\Delta}} : H^{s-1/2}(Y, (E^1 \oplus E^2)_{|Y}) \longrightarrow \mathrm{Ker}(\widehat{\Delta}, s) \subset H^s(X, E^1 \oplus E^2), \tag{4.10}$$

and Calderon projector

$$P(\widehat{\Delta}) = \gamma_0 \widehat{\mathcal{K}} \ .$$

We can compute $P(\widehat{\Delta})$ quite explicitly:

**Lemma 4.4.**

$$P(\widehat{\Delta}) = \begin{pmatrix} P(D) & \gamma D_{P(D)}^{-1}\mathcal{K}_* \\ 0 & P(D^*) \end{pmatrix} \ , \tag{4.11}$$

*where $\mathcal{K}_*$ is the Poisson operator for $D^*$.*

We postpone the proof for the moment. Notice, however, since $\widehat{R} \neq 0$, that $P(\widehat{\Delta}) - P(D) \oplus P(D^*)$ is only a $\psi$do of order $-1$ and not smoothing due to the off-diagonal term. Further, it is a projector but not a projection (cf. Remark 4.3(4)).

Since $\widehat{\Delta}$ is of Dirac-type, we have a $\psi$do Grassmannian $Gr_1(\widehat{\Delta})$ of global boundary conditions for $\widehat{\Delta}$, and for each $Q \in Gr_1(\widehat{\Delta})$ a first-order global boundary problem

$$\widehat{\Delta}_Q = \widehat{\Delta} : \mathrm{dom}\,(\widehat{\Delta}_Q) \to L^2(X, E^1 \oplus E^2)\ .$$

We recover the resolvent $(\Delta_P - \lambda)^{-1}$ in the following way. First, we have a canonical map

$$Gr_\infty^{(r)}(D) \longrightarrow Gr_1^{(0)}(\widehat{\Delta})\ ,$$
$$P \longmapsto \widehat{P} := P \oplus P^\star\ .$$

To see that $\widehat{P}$ is in the index zero component $Gr_1^{(0)}(\widehat{\Delta})$, observe that the identity $I : \widehat{H}_Y \to \widehat{H}_Y$ acting between the block decompositions $H(D) \oplus H(D)^\perp$, $W \oplus W^\perp$, where $W = \mathrm{ran}\,(P)$, of $\widehat{H}_Y = L^2(Y, (E^1 \oplus E^2)_{|Y})$ is

$$I = \begin{pmatrix} S(P) & S^\perp(P) \\ S(P^\perp) & S^\perp(P^\perp) \end{pmatrix}\ . \tag{4.12}$$

For any $\psi$do $B$ on $H_Y$ set

$$S^\perp(B) = B \circ P(D)^\perp : H(D)^\perp \longrightarrow \mathrm{ran}\,(B)\ ,$$

and for $P \in Gr_\infty(D)$ note that

$$S^\star(P^\star) := P^\star \circ P(D^*) = \sigma S^\perp(P^\perp)\sigma^{-1} : \sigma H(D)^\perp \longrightarrow \sigma W\ , \tag{4.13}$$

and let

$$S(\widehat{P}) = \widehat{P} \circ P(\widehat{\Delta}) : H(\widehat{\Delta}) \to \mathrm{ran}\,(\widehat{P}) = W \oplus W^\perp\ ,$$

where we use (3.15). Then from (4.11) and (4.12)

$$\mathrm{ind}\,(S(\widehat{P})) = \mathrm{ind}\,(S(P)) + \mathrm{ind}\,(S^\perp(P^\perp)) = \mathrm{ind}\,\left(I - \begin{pmatrix} 0 & S^\perp(P) \\ S(P^\perp) & 0 \end{pmatrix}\right)$$

which is zero, since the matrix operator is a $\psi$do of order $-1$ and hence compact. Alternatively, this fact follows from $\mathrm{ind}\,(S(\widehat{P})) = \mathrm{ind}\,(\widehat{\Delta}_{\widehat{P}})$ and the identity

$$\sigma \widehat{P}^\perp \sigma^{-1} = \widehat{P}\ ,$$

which along with (4.8), (4.9) implies that $\widehat{\Delta}_{\widehat{P}}$ is self-adjoint considered as a closed operator on $L^2(X, E^1 \oplus E^2)$.

Next, we have a canonical inclusion defined by $D$

$$\widehat{\imath} : H^1(X, E^1) \longrightarrow L^2(X; E^1 \oplus E^2)\ , \qquad \widehat{\imath}(\psi) = (\psi, D\psi)\ .$$

Setting for $\lambda \in \mathbb{C}$

$$\widehat{\Delta}_\lambda = \begin{pmatrix} -\lambda & D^* \\ D & -I \end{pmatrix} : H^1(X; E^1 \oplus E^2) \to L^2(X; E^1 \oplus E^2)\ ,$$

the inclusion $\widehat{i}$ restricts to an isomorphism

$$\widehat{i}_{|\mathrm{Ker}} : \mathrm{Ker}(\Delta-\lambda) \xrightarrow{\simeq} \mathrm{Ker}(\widehat{\Delta}_\lambda) \subset H^2(X,E^1)\oplus H^1(X,E^2) \ ,$$

with inverse $(s_1,s_2)\mapsto s_1$, where $\Delta-\lambda, \widehat{\Delta}_\lambda$ are acting in $H^2(X,E^1), H^1(X,E^1\oplus E^2)$, respectively. That $\widehat{i}_{|\mathrm{Ker}}$ is injective with range $\mathrm{Ker}(\widehat{\Delta}_\lambda)$ follows from the identity

$$\widehat{\Delta}_\lambda \begin{pmatrix} \psi \\ D\psi \end{pmatrix} = \begin{pmatrix} (\Delta-\lambda)\psi \\ 0 \end{pmatrix} \ . \tag{4.14}$$

On the other hand, if $(s_1,s_2)\in \mathrm{Ker}(\widehat{\Delta}_\lambda)$, then $D^*s_2=\lambda s_1$ and $s_2=Ds_1$ and hence $s_1\in\mathrm{Ker}(\Delta-\lambda)$. In particular, setting $s_i=\widehat{i}(\psi_i)$ we can extract Green's formula for $\Delta$ from (4.9) and (4.14) (with $\lambda=0$) :

$$<\Delta\psi_1,\psi_2> - <\psi_1,\Delta\psi_2> = \left\langle -\widehat{\sigma}\gamma_0 \begin{pmatrix} \psi_1 \\ D\psi_1 \end{pmatrix}, \gamma_0 \begin{pmatrix} \psi_2 \\ D\psi_2 \end{pmatrix} \right\rangle \ .$$

The operator $\widehat{i}$ also restricts to a canonical inclusion

$$\widehat{i} : \mathrm{dom}\,(\Delta_P) \longrightarrow \mathrm{dom}\,(\widehat{\Delta}_{\widehat{P}}) \ . \tag{4.15}$$

From (4.14) and (4.15) we have for $\lambda\in\mathbb{C}\backslash\overline{\mathbb{R}}_+$

$$(\Delta_P-\lambda)^{-1} = \left[\widehat{\Delta}^{-1}_{\lambda,\widehat{P}}\right]_{(1,1)} : L^2(X,E^1) \longrightarrow \mathrm{dom}\,(\Delta_P-\lambda) \ , \tag{4.16}$$

where for an operator $C=\begin{pmatrix} S & T \\ U & V\end{pmatrix}$ on $L^2(X,E^1\oplus E^2)$, we define $[C]_{(1,1)}=S$. Equivalently,

$$(\Delta_P-\lambda)^{-1} = \begin{pmatrix} I & 0 \end{pmatrix} \widehat{\Delta}^{-1}_{\lambda,\widehat{P}} \begin{pmatrix} I \\ 0 \end{pmatrix} \ . \tag{4.17}$$

A precise formula for $\widehat{\Delta}^{-1}_{\lambda,\widehat{P}}$ is given in (4.36).

- **The scattering determinant**

Let $P_1,P_2\in Gr_\infty(D)$ and for $\mu\in\mathbb{C}\setminus\overline{\mathbb{R}}_+$ set

$$S_\mu(\widehat{P}_i) := \widehat{P}_i\circ P(\widehat{\Delta}_\mu) : H(\widehat{\Delta}_\mu) \longrightarrow \mathrm{ran}\,(\widehat{P}_i) \ .$$

Let $\mathcal{E} : \mathrm{ran}\,(P_2) \to \mathrm{ran}\,(P_1), \tilde{\mathcal{E}} : \mathrm{ran}\,(P_2^\star) \to \mathrm{ran}\,(P_1^\star)$ be auxiliary operators for $D_{P_i}, D^*_{P_i}$ respectively. Then

$$\widehat{\mathcal{E}} = \begin{pmatrix} \mathcal{E} & 0 \\ 0 & \tilde{\mathcal{E}} \end{pmatrix} : \mathrm{ran}\,(\widehat{P}_2) \to \mathrm{ran}\,(\widehat{P}_1) \tag{4.18}$$

is an auxiliary operator for $\widehat{\Delta}_{\widehat{P}_1}, \widehat{\Delta}_{\widehat{P}_2}$ and we have:

**Proposition 4.5.** $(\Delta_{P_1}, \Delta_{P_2})$ *are $\zeta$-comparable with scattering determinant*

$$\det_{\widehat{\mathcal{E}}}(\widehat{\Delta}_{\mu,\widehat{P}_1}, \widehat{\Delta}_{\mu,\widehat{P}_2}) = \det_F((\widehat{\mathcal{E}} S_\mu(\widehat{P}_2))^{-1} S_\mu(\widehat{P}_1))$$

*taken on $H(\widehat{\Delta}_\mu)$. With $\theta = \pi$ and $\lambda \in \mathbb{R}_+$, one has*

$$\frac{\det_\zeta(\Delta_{P_1})}{\det_\zeta(\Delta_{P_2})} = \det_F\left(\frac{S(\widehat{P}_1)}{\widehat{\mathcal{E}}S(\widehat{P}_2)}\right) . \, e^{-\mathrm{LIM}_{\lambda\to+\infty}\log\det_F((\widehat{\mathcal{E}}S_{-\lambda}(\widehat{P}_2))^{-1}S_{-\lambda}(\widehat{P}_1))} \, . \tag{4.19}$$

*If $(P_2, P_1)$ is invertible*

$$\frac{\det_\zeta(\Delta_{P_1})}{\det_\zeta(\Delta_{P_2})} = \det_F\left(\frac{S(\widehat{P}_1)}{\widehat{P}_1 S(\widehat{P}_2)}\right) . \, e^{-\mathrm{LIM}_{\lambda\to+\infty}\log\det_F((\widehat{P}_1 S_{-\lambda}(\widehat{P}_2))^{-1}S_{-\lambda}(\widehat{P}_1))} \, . \tag{4.20}$$

*Proof.* From [10] Cor(9.5) ,[11] Thm(1), the coefficients $a_{j,k}, a_j$ in the asymptotic expansion (4.2) are locally determined by the symbols of $\Delta$ and $B$, while, provided $P_1 - P_2 \in \Psi_l(H_Y)$ with $l \geq n$, the expansion coefficients differ only in the $c_j$. Integrating we hence obtain a resolvent trace expansion in closed subsectors of $\mathbb{C}\backslash\overline{\mathbb{R}}_+$

$$\mathrm{Tr}\,((\Delta_{P_1} - \mu)^{-1} - (\Delta_{P_2} - \mu)^{-1}) \sim \tag{4.21}$$

$$\sum_{j=1}^{\infty}\sum_{k=0}^{1} C_{j,k}(-\mu)^{-j/2-1}\log(-\mu) + \zeta(\Delta_{P_1}, \Delta_{P_2}, 0)(-\mu)^{-1}$$

where the coefficients $C_{j,k} = C_{j,k}(\Delta, P_1, P_2)$ differ from the $c_j$ by universal constants.

Since $\widehat{P}_1 - \widehat{P}_2$ has a smooth kernel we know from (3.36) that so does $\widehat{\Delta}^{-1}_{\lambda,\widehat{P}_1} - \widehat{\Delta}^{-1}_{\lambda,\widehat{P}_2}$, and from (4.16) also $\Delta^{-1}_{P_1} - \Delta^{-1}_{P_2}$. From (4.16), (4.17) we have

$$\begin{aligned}
\mathrm{Tr}\,\left((\Delta_{P_1} - \mu)^{-1} - (\Delta_{P_2} - \mu)^{-1}\right) &= \mathrm{Tr}\left(\left[\widehat{\Delta}^{-1}_{\mu,\widehat{P}_1} - \widehat{\Delta}^{-1}_{\mu,\widehat{P}_2}\right]_{(1,1)}\right) \\
&= \mathrm{Tr}\left(\begin{pmatrix} I & 0 \end{pmatrix}(\widehat{\Delta}^{-1}_{\mu,\widehat{P}_1} - \widehat{\Delta}^{-1}_{\mu,\widehat{P}_2})\begin{pmatrix} I \\ 0 \end{pmatrix}\right) \\
&= \mathrm{Tr}\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}(\widehat{\Delta}^{-1}_{\mu,\widehat{P}_1} - \widehat{\Delta}^{-1}_{\mu,\widehat{P}_2})\right) \\
&= -\mathrm{Tr}\left(\frac{\partial}{\partial\mu}(\widehat{\Delta}_\mu)\left(\widehat{\Delta}^{-1}_{\mu,\widehat{P}_1} - \widehat{\Delta}^{-1}_{\mu,\widehat{P}_2}\right)\right) \\
&= -\frac{\partial}{\partial\mu}\log\det_F\left(\frac{S_\mu(\widehat{P}_1)}{\widehat{\mathcal{E}}S_\mu(\widehat{P}_2)}\right) \, ,
\end{aligned}$$

where we use (3.32) for the final equality, since the variation in $U$ is of order 0.

Hence $(\Delta_{P_1}, \Delta_{P_2})$ are $\zeta$-comparable. Since they are also $\zeta$-admissible, (4.19) is a consequence of Theorem 2.5. □

• **Relation to the right-side of** (4.4)

**Proposition 4.6.** *Let $S_r(P_i)$ be the boundary integrals defined by a smooth 1-parameter family of Dirac-type operators $D_r$. Then*

$$\frac{d}{dr}\log\det{}_F\left(\frac{S_r(\widehat{P}_1)}{\widehat{\mathcal{E}}S_r(\widehat{P}_2)}\right)=\frac{d}{dr}\log\frac{\det_F\left(S_r(P_1)^*S_r(P_1)\right)}{\det_F\left(S_r(P_2)^*S_r(P_2)\right)}\ . \tag{4.22}$$

*If* $(P_2,P_1):\mathrm{ran}\,(P_2)\to\mathrm{ran}\,(P_1)$ *is invertible, one has*

$$\det{}_F\left(\frac{S(\widehat{P}_1)}{\widehat{P}_1S(\widehat{P}_2)}\right)=\frac{\det_F\left(S(P_1)^*S(P_1)\right)}{\det_F\left(S(P_2)^*S(P_2)\right)}\cdot\frac{1}{\det_F\left(P_2P_1P_2\right)}\ , \tag{4.23}$$

*the determinant in the denominator being taken on* $\mathrm{ran}\,(P_2)$*.*

*Proof.* Equation (4.23) is a consequence of the following identity.

**Lemma 4.7.**

$$\det{}_F\left(\frac{S^{\perp}(P_1^{\perp})}{P_1^{\perp}S^{\perp}(P_2^{\perp})}\right)=\overline{\det{}_F\left(\frac{S(P_1)}{P_1S(P_2)}\right)}\ , \tag{4.24}$$

*where the left-hand determinant is taken on* $H(D)^{\perp}$*.*

*Proof.* As in Proposition 3.16, we may choose graph Stiefel coordinates

$$H(D)=\mathrm{graph}(K:E\to E^{\perp})\ ,\qquad W_i=\mathrm{graph}(T_i:E\to E^{\perp})\ . \tag{4.25}$$

Using the property

$$\overline{\det{}_F A}=\det{}_F A^* \tag{4.26}$$

for $A:E\to E$ of determinant class and letting $\det_{[E]}(A)$ mean the Fredholm determinant taken on $E$, we have from (3.56)

$$\overline{\det{}_F\left(\frac{S(P_1)}{P_1S(P_2)}\right)}=\frac{\det_{[E]}(I+K^*T_1)}{\det_{[E]}(I+K^*T_2)}\frac{\det_{[E]}(I+T_2^*T_2)}{\det_{[E]}(I+T_2^*T_1)}\ .$$

From (4.25) we have

$$H(D)^{\perp}=\mathrm{graph}(-K^*:E^{\perp}\to E)\ ,\qquad W_i^{\perp}=\mathrm{graph}(-T_i^*:E^{\perp}\to E)\ ,$$

and so in Stiefel coordinates

$$P(D)^{\perp}=\begin{pmatrix}K^*\widehat{Q}_K^{-1}K & -K^*\widehat{Q}_K^{-1}\\ -\widehat{Q}_K^{-1}K & \widehat{Q}_K^{-1}\end{pmatrix}\ ,\qquad P_i^{\perp}=\begin{pmatrix}T_i^*\widehat{Q}_i^{-1}T_i & -T_i^*\widehat{Q}_i^{-1}\\ -\widehat{Q}_i^{-1}T_i & \widehat{Q}_i^{-1}\end{pmatrix}\ , \tag{4.27}$$

where $\widehat{Q}_K=I+KK^*$, $\widehat{Q}_i=I+T_iT_i^*$. Using these local representations we compute in a similar fashion to (3.56)

$$\det{}_F\left(\frac{S^{\perp}(P_1^{\perp})}{P_1^{\perp}S^{\perp}(P_2^{\perp})}\right)=\frac{\det_{[E^{\perp}]}(I+T_1K^*)}{\det_{[E^{\perp}]}(I+T_1K^*)}\frac{\det_{[E^{\perp}]}(I+T_2T_2^*)}{\det_{[E^{\perp}]}(I+T_1T_2^*)}\ .$$

Since $\det_{[E^{\perp}]}(I+ST^*)=\det_{[E]}(I+T^*S)$ for any $S,T:E\to E^{\perp}$ of trace class, we reach the conclusion. $\square$

From Lemma 4.4

$$S(\widehat{P}_1) = \begin{pmatrix} P_1 & 0 \\ 0 & P_1^* \end{pmatrix} \begin{pmatrix} P(D) & \gamma_0 D^{-1}_{P(D)}\mathcal{K}_* \\ 0 & P(D^*) \end{pmatrix} = \begin{pmatrix} S(P_1) & P_1\gamma_0 D^{-1}_{P(D)}\mathcal{K}_* \\ 0 & \sigma S^{\perp}(P_1^{\perp})\sigma^{-1} \end{pmatrix} ,$$

where we use (4.13). Computing $\widehat{P}_1 S(\widehat{P}_2)$ in a similar way, and using (4.24), (4.26) and the multiplicativity of the Fredholm determinant, we obtain

$$\det_F \left( \frac{S(\widehat{P}_1)}{\widehat{P}_1 S(\widehat{P}_2)} \right) = \det_F \left( \frac{S(P_1)}{P_1 S(P_2)} \right) . \det_F \left( \frac{S^{\perp}(P_1^{\perp})}{P_1^{\perp} S^{\perp}(P_2^{\perp})} \right) \tag{4.28}$$

$$\begin{aligned}
&= \det_F \left( \left( \frac{S(P_1)}{P_1 S(P_2)} \right)^* \frac{S(P_1)}{P_1 S(P_2)} \right) \\
&= \det_F \left( \frac{S(P_1)^* S(P_1)}{S(P_2)^* P_2 P_1 P_2 S(P_2)} \right) \\
&= \frac{\det_F \left( S(P_1)^* S(P_1) \right)}{\det_F \left( S(P_2)^* S(P_2) \right)} \cdot \frac{1}{\det_F \left( [S(P_2)^* S(P_2)]^{-1} S(P_2)^* P_2 P_1 P_2 S(P_2) \right)} \\
&= \frac{\det_F \left( S(P_1)^* S(P_1) \right)}{\det_F \left( S(P_2)^* S(P_2) \right)} \cdot \frac{1}{\det_F (P_2 P_1 P_2)} ,
\end{aligned}$$

which proves (4.23).

To see (4.22), first note that in the same way as (4.28) we have

$$\det_F \left( \frac{S_r(\widehat{P}_1)}{\widehat{\mathcal{E}} S_r(\widehat{P}_2)} \right) = \det_F \left( \frac{S_r(P_1)}{\mathcal{E} S_r(P_2)} \right) . \det_F \left( \frac{S_r^{\perp}(P_1^{\perp})}{\mathcal{E}^{\perp} S_r^{\perp}(P_2^{\perp})} \right) , \tag{4.29}$$

where $\mathcal{E}^{\perp} := \sigma\tilde{\mathcal{E}}\sigma^{-1}$. Thus we have to show that

$$\sum_{i=1}^{2} \operatorname{Tr} \left( S_r^{\perp}(P_i^{\perp})^{-1} \frac{d}{dr} S_r^{\perp}(P_i^{\perp}) \right) = \sum_{i=1}^{2} \overline{\operatorname{Tr}} \left( S_r(P_i)^{-1} \frac{d}{dr} S_r(P_i) \right) ,$$

where the right-side means complex conjugate. This follows by the same method used in Lemma 4.7, via the Stiefel coordinate representation for $S(P_i)^{-1}$ (use (3.60) with $T = K, \widehat{T} = T_i$) and its analogue for $S^{\perp}(P_i^{\perp})^{-1}$ (use (4.27) ). Or, these coordinate matrices may be used to prove directly, in a similar way to Proposition 3.16, that (4.29) differs from the right-side of (4.4) by a function independent of $P(D)$. □

**Proposition 4.8.** *Let $D_r$, $-\varepsilon \le r \le \varepsilon$, be a 1-parameter family of Dirac-type operators with product case geometry such that $\widehat{\Delta}(r) = \begin{pmatrix} 0 & D_r^* \\ D_r & -I \end{pmatrix}$ satisfies (3.30). Then $P_1, P_2 \in Gr_\infty(D_0)$ are global boundary conditions for $D_r$. If the $D_{r,P_i}$ are invertible, then, with*

$$S_r(P_i) = P \circ P(D_r) : H(D_r) \to \operatorname{ran}(P) ,$$

*one has*

$$\frac{d}{dr} \log \frac{\det_\zeta(\Delta_{r,P_1})}{\det_\zeta(\Delta_{r,P_2})} = \frac{d}{dr} \log \frac{\det_F \left( S_r(P_1)^* S_r(P_1) \right)}{\det_F \left( S_r(P_2)^* S_r(P_2) \right)} . \tag{4.30}$$

*Proof.* Let $S_r(\widehat{P}) = \widehat{P} \circ P(\widehat{\Delta}(r)) : H(\widehat{\Delta}(r)) \to \operatorname{ran} \widehat{P}$. Then from Proposition 3.32 we have

$$\operatorname{Tr}\left(\frac{d}{dr}(\widehat{\Delta}(r))\left(\widehat{\Delta}(r)^{-1}_{\widehat{P}_1} - \widehat{\Delta}(r)^{-1}_{\widehat{P}_2}\right)\right) = \frac{d}{dr}\log\det{}_F\left(\frac{S_r(\widehat{P}_1)}{\widehat{\mathcal{E}}S_r(\widehat{P}_2)}\right) . \tag{4.31}$$

In view of (4.22), we need to prove that left-side (4.30) = left-side (4.31). We show each expression is equal to

$$\operatorname{Tr}_{L^2}\left(\dot{D}(D^{-1}_{P_1} - D^{-1}_{P_2})\right) + \operatorname{Tr}_{L^2}\left(\dot{D}^*((D^*_{P_1})^{-1} - (D^*_{P_2})^{-1})\right) , \tag{4.32}$$

where $\dot{D} = (d/dr)D_r$. We omit $r$ from the operator notation throughout.

First, for any $P, P_1, P_2 \in \operatorname{Gr}_\infty(D)$ we record the following identities:

$$\dot{\Delta} = \dot{D}^*D + D^*\dot{D} , \tag{4.33}$$

$$D\Delta_P^{-1} = (\Delta_P^*)^{-1} , \qquad D^*\tilde{\Delta}_P^{-1} = D_P^{-1}\,(\Delta_P^*)^{-1} , \tag{4.34}$$

$$(D^*_{P_1})^{-1}\Delta^{-1}_{P_2} = \tilde{\Delta}^{-1}_{P_2}(D^*_{P_1})^{-1} , \tag{4.35}$$

$$\widehat{\Delta}^{-1}_{\lambda,\hat{P}} = \begin{pmatrix} (\Delta_P - \lambda)^{-1} & D_P^*(\tilde{\Delta}_P - \lambda)^{-1} \\ D_P(\Delta_P - \lambda)^{-1} & \lambda(\tilde{\Delta}_P - \lambda)^{-1} \end{pmatrix} , \tag{4.36}$$

where $\tilde{\Delta} = D^*D$, $\tilde{\Delta}_P = D_P D_P^*$. To see (4.34), since $D^*D\Delta_P^{-1} = I$ on $L^2(X,E)$, and $D\Delta_P^{-1}$ has range in $\operatorname{dom}(D_P^*)$, one has (using (3.18) for $D_P^*$)

$$(D_P^*)^{-1} = \left((D_P^*)^{-1}D^*\right)D\Delta_P^{-1} = D\Delta_P^{-1} - \mathcal{K}_*(P^\star)\gamma_0 D\Delta_P^{-1} = D\Delta_P^{-1} .$$

The other identities can be checked in a similar fashion. For brevity let $\Delta_i = \Delta_{P_i}, D_i = D_{P_i}, D_i^* = D^*_{P_i} := D^*_{P_i^\star}$.

Setting $\lambda = 0$ in (4.36) we have

$$\dot{\widehat{\Delta}} = \begin{pmatrix} 0 & \dot{D}^* \\ \dot{D} & 0 \end{pmatrix} , \qquad \widehat{\Delta}^{-1}_{\widehat{P}_i} = \begin{pmatrix} \Delta_i^{-1} & D_i^{-1} \\ (D_i^*)^{-1} & 0 \end{pmatrix} ,$$

and hence $\dot{\widehat{\Delta}}\left(\widehat{\Delta}^{-1}_{\widehat{P}_1} - \widehat{\Delta}^{-1}_{\widehat{P}_2}\right) = \begin{pmatrix} \dot{D}^*\left((D_1^*)^{-1} - (D_2^*)^{-1}\right) & 0 \\ \dot{D}\left(\Delta_1^{-1} - \Delta_2^{-1}\right) & \dot{D}\left(D_1^{-1} - D_2^{-1}\right) \end{pmatrix}$, from which the equality of the left-side of (4.31) with (4.32) is clear.

Next, let $\zeta_{rel}(0) = \zeta(\Delta_1, \Delta_2, 0)$. The resolvent trace (4.21) implies a heat trace expansion as $t \to 0$

$$\operatorname{Tr}(e^{-t\Delta_1} - e^{-t\Delta_2})) \sim \sum_{j=1}^{\infty}\sum_{k=0}^{1}\tilde{C}_{j,k}t^{j/2}\log^k t + \zeta_{rel}(0) , \tag{4.37}$$

while from (2.46) we have

$$\log\det_{\zeta,\theta}(\Delta_1,\Delta_2) = \left[\int_0^\infty t^{s-1}\mathrm{Tr}\left(e^{-t\Delta_1}-e^{-t\Delta_2}\right)\,dt - \frac{\zeta_{rel}(0)}{s}\right]_{|s=0} - \gamma\zeta_{rel}(0)\ .$$

Since $\Delta_1^{-1}-\Delta_2^{-1}$ has a smooth kernel, precisely the same argument as that leading to (3.43) yields $(d/dr)\zeta_{rel}(0)=0$. So the $r$-variation 'kills' the pole at $s=0$. From (4.37) we therefore have $(d/dr)\mathrm{Tr}\,(e^{-t\Delta_1}-e^{-t\Delta_2}))=O(t^{1/2})$, and hence

$$\begin{aligned}\frac{d}{dr}\log\det_{\zeta,\theta}(\Delta_1,\Delta_2) &= -\int_0^\infty t^{-1}\frac{d}{dr}\left[\mathrm{Tr}\,(e^{-t\Delta_1})-\mathrm{Tr}\,(e^{-t\Delta_2})\right]\,dt \\ &= \int_0^\infty \mathrm{Tr}\,(\dot{\Delta}_1e^{-t\Delta_1}-\dot{\Delta}_2e^{-t\Delta_2})\,dt\end{aligned} \tag{4.38}$$

$$= \int_0^\infty \mathrm{Tr}\,(\dot{D}^*D(e^{-t\Delta_1}-e^{-t\Delta_2}) \ + \ \mathrm{Tr}\,(D^*\dot{D}(e^{-t\Delta_1}-e^{-t\Delta_2})\,dt\ ,$$

where in the second equality we use Duhamel's Formula and the symmetry of the trace. The heat operator

$$e^{-t\Delta_i} = \frac{i}{2\pi}\int_{C_\pi} e^{-t\lambda}(\Delta_i-\lambda)^{-1}\,d\lambda : L^2(X,E)\longrightarrow \mathrm{dom}\,(\Delta_i)$$

has range in $\mathrm{dom}\,(\Delta_i)$, and hence it follows that $D^*\dot{D}e^{-t\Delta_i}=D_i^*\dot{D}_ie^{-t\Delta_i}$, since $P_i\gamma_0\psi=0, P_i^\star\gamma_0 D\psi=0$ implies the domain $P_i\gamma_0\psi=0, P_i^\star\gamma_0\dot{D}\psi=0$ for $D^*\dot{D}$. Thus, using also (4.35) and the contour integral definition of $e^{-t\tilde{\Delta}_i}$, which imply $e^{-t\Delta_i}D_i^*=D_i^*e^{-t\tilde{\Delta}_i}$, we have

$$\begin{aligned}\mathrm{Tr}\,\left(D^*\dot{D}e^{-t\Delta_i}\right) &= \mathrm{Tr}\,\left(D_i^*\dot{D}_ie^{-t\Delta_i}\right) = \mathrm{Tr}\,\left(D_i^*\dot{D}_ie^{-t\Delta_i}D_i^*(D_i^*)^{-1}\right) \\ &= \mathrm{Tr}\,\left(D_i^*\dot{D}_iD_i^*e^{-t\tilde{\Delta}_i}(D_i^*)^{-1}\right) = \mathrm{Tr}\,\left(\dot{D}D^*e^{-t\tilde{\Delta}_i}\right)\ .\end{aligned} \tag{4.39}$$

Hence equation (4.38) equals

$$\begin{aligned}&\int_0^\infty \mathrm{Tr}\,\left(\dot{D}^*D(e^{-t\Delta_1}-e^{-t\Delta_2})\right)+\mathrm{Tr}\,\left(\dot{D}D^*(e^{-t\tilde{\Delta}_1}-e^{-t\tilde{\Delta}_2})\right)\,dt \\ =\ & -\int_0^\infty \frac{\partial}{\partial t}\mathrm{Tr}\,\left(\dot{D}^*D(\Delta_1^{-1}e^{-t\Delta_1}-\Delta_2^{-1}e^{-t\Delta_2})\right) \\ &\qquad -\int_0^\infty -\frac{\partial}{\partial t}\mathrm{Tr}\,\left(\dot{D}D^*(\tilde{\Delta}_1^{-1}e^{-t\tilde{\Delta}_1}-\tilde{\Delta}_2^{-1}e^{-t\tilde{\Delta}_2})\right)\,dt \\ =\ & -\lim_{\varepsilon\to 0}\mathrm{Tr}\,\left(\dot{D}^*D(\Delta_1^{-1}e^{-t\Delta_1}-\Delta_2^{-1}e^{-t\Delta_2})\right)|_\varepsilon^{1/\varepsilon} \\ &\qquad -\lim_{\varepsilon\to 0}\mathrm{Tr}\,\left(\dot{D}D^*(\tilde{\Delta}_1^{-1}e^{-t\tilde{\Delta}_1}-\tilde{\Delta}_2^{-1}e^{-t\tilde{\Delta}_2})\right)|_\varepsilon^{1/\varepsilon} \\ =\ & \mathrm{Tr}\,\left(\dot{D}^*D(\Delta_1^{-1}-\Delta_2^{-1})\right)+\mathrm{Tr}\,\left(\dot{D}D^*(\tilde{\Delta}_1^{-1}-\tilde{\Delta}_2^{-1}\right) \\ =\ & \mathrm{Tr}\,\left(\dot{D}^*((D_1^*)^{-1}-(D_2^*)^{-1})\right)+\mathrm{Tr}\,\left(\dot{D}(D_1^{-1}-D_2^{-1}\right)\ ,\end{aligned}$$

where we use (4.34) for the final equality, and this completes the proof. □

**Remark 4.9.** *The variational equality* (4.30) *also follows from* (3.31) *applied to* (4.32)*, along with an analogue of Proposition 4.6.*

**Corollary 4.10.** *For $P_i \in Gr_\infty(D)$ with $D_{P_i}$ invertible*

$$\frac{\det_\zeta(\Delta_{P_1})}{\det_\zeta(\Delta_{P_2})} = \frac{\det_F\left(S(P_1)^*S(P_1)\right)}{\det_F\left(S(P_2)^*S(P_2)\right)}.N(P_1,P_2)\ , \tag{4.40}$$

*where $N(P_1,P_2)$ depends only the boundary data. One has*

$$N(P_1,P_2).N(P_2,P_3) = N(P_1,P_3)\ . \tag{4.41}$$

Integrating (4.30) over $[0,t] \subset (-\varepsilon,\varepsilon)$, (4.40) can be restated

$$\frac{\det_\zeta(\Delta_{t,P_1})}{\det_\zeta(\Delta_{t,P_2})}.\frac{\det_\zeta(\Delta_{0,P_2})}{\det_\zeta(\Delta_{0,P_1})} = \frac{\det_F\left(S_t(P_1)^*S_t(P_1)\right)}{\det_F\left(S_t(P_2)^*S_t(P_2)\right)}.\frac{\det_F\left(S_0(P_2)^*S_0(P_2)\right)}{\det_F\left(S_0(P_1)^*S_0(P_1)\right)}\ . \tag{4.42}$$

Next, we make use of the homogeneous structure of the Grassmannian to prove

- $N(P_1,P_2) = 1$.

We use a variational argument generalizing [28]. Let

$$U_\infty(H_Y) = U(H_Y) \cap (I + \Psi_\infty(H_Y))$$

be the group of unitary operators on $H_Y = L^2(Y,(E_1)_{|Y})$ differing from the identity by a smoothing operator, and let $\tilde{Gr}_\infty(D)$ be the dense open subset of the index zero component of $Gr_\infty(D)$

$$\begin{aligned} \tilde{Gr}_\infty(D) &= \{P \in Gr_\infty(D) \mid D_P \text{ invertible}\} \\ &= \{P \in Gr_\infty(D) \mid S(P) : H(D) \to \mathrm{ran}\,(P) \text{ invertible}\} \\ &= U_{H(D)}\ , \end{aligned} \tag{4.43}$$

where the final equality refers to (3.50).

**Lemma 4.11.** *For any $P_1, P_2 \in \tilde{Gr}_\infty(D)$ there exists a smooth path*

$$I = g_0 \le g_r \le g_1 = g\ , \qquad\qquad 0 \le r \le 1\ , \tag{4.44}$$

*in $U_\infty(H_Y)$, defining smooth paths of projections $P_{1,r} = g_r P_1 g_r^{-1}$ and $P_{2,r} = g_r P_2 g_r^{-1}$ in $\tilde{Gr}_\infty(D)$ with $gP_1g^{-1} = P_2$,*

$$P_1 \le P_{1,r} \le P_2\ , \tag{4.45}$$

*and*

$$gP_1g^{-1} \le P_{2,r} \le gP_2g^{-1}\ . \tag{4.46}$$

We hence obtain a real-valued strictly positive function $g_r \longmapsto N(P_{1,r},P_{2,r})$. The decisive fact is the following:

**Lemma 4.12.**

$$\frac{d}{dr}\log N(P_{1,r},P_{2,r}) = 0\ . \tag{4.47}$$

The proofs will be given in a moment. Integrating (4.47) we have

$$N(P_{1,1},P_{2,1}) = N(P_{1,0},P_{2,0}) \ . \tag{4.48}$$

From (4.45), (4.46), (4.48) we obtain

$$N(P_2, gP_2g^{-1}) = N(P_1, gP_1g^{-1}) \ ,$$

and hence that $N(P, gPg^{-1})$ depends only on $g \in U_\infty$ and not on the basepoint $P$. We define

$$N(g) := N(P, gPg^{-1})$$

where $P, gPg^{-1} \in \tilde{Gr}_\infty(D)$. Then for $g_1, g_2 \in U_\infty(H_Y)$, from (4.41) we have with $P, g_2Pg_2^{-1}, g_1g_2Pg_2^{-1}g_1^{-1} \in \tilde{Gr}_\infty(D)$,

$$\begin{aligned} N(g_1g_2) &= N(P, g_1g_2Pg_2^{-1}g_1^{-1}) \\ &= N(P, g_2Pg_2^{-1}).N(g_2Pg_2^{-1}, g_1(g_2Pg_2^{-1})g_1^{-1}) \\ &= N(g_1).N(g_2) \ . \end{aligned}$$

Thus $g \mapsto N(g)$ extends to a (Banach) character on $U_\infty(H_Y)$. It is a well-known and elementary fact that the only such characters on $U_\infty(H_Y)$ are $g \mapsto \det_F(g)$ , $g \mapsto \det_F(g^{-1})$ or the trivial character $g \mapsto 1$. But $N$ is real-valued positive, while $\det_F$ on $U_\infty(H_Y)$ takes values in $U(1)$. Hence $N(g) = 1$.

This completes the proof of Theorem 4.2. It remains to prove the above Lemmas.

**Proof of Lemma 4.11** First, we have from (4.43) that $\tilde{Gr}_\infty(D)$ is path connected, and in fact contractible. To show that a path of the asserted form exists we prove that $U_\infty(H_Y)$ acts transitively on the *index zero* component of $Gr_\infty(D)$, with non-contractible stabilizer subgroup $U_\infty(W) \times U_\infty(W^\perp)$ at $P \in Gr_\infty^{(0)}(D)$, $\mathrm{ran}\,(P) = W$. (The global homogeneous structure on $Gr_\infty(D)$ is usually studied via the action of a restricted linear group [22], with contractible stabilizer $U(W) \times U(W^\perp)$, but our purposes on $Gr_\infty^{(0)}(D)$ are better suited to the $U_\infty(H_Y)$ subgroup action.)

It is enough to give the path (4.44) in $GL_\infty(H_Y) = GL(H_Y) \cap (I + \Psi_\infty(H_Y))$, the group of invertibles congruent to the identity. For $U_\infty(H_Y)$ is a retraction of $GL_\infty(H_Y)$ via the phase map

$$GL_\infty(H_Y) \longrightarrow U_\infty(H_Y) \ , \qquad g \longmapsto u_g = g|g|^{-1} \ ,$$

where $|g| := (g^*g)^{1/2}$. Here

$$(g^*g)^t = \frac{i}{2\pi}\int_\gamma \mu^t (g^*g - \mu)^{-1} d\mu \ , \tag{4.49}$$

with $\gamma$ a contour surrounding $\mathrm{sp}(g^*g)$, is a smooth map $\mathbb{C} \times GL_\infty(H_Y) \to GL_\infty(H_Y)$ (Lemma(7.10) [23]). It follows that if $g_r$ is a path in $GL_\infty(H_Y)$ satisfying the properties of Lemma 4.11 apart from unitarity, then $u_{g_r}$ will be the path required. To see this, if $P_2 = gP_1g^{-1}$ with $g \in GL_\infty(H_Y)$, then, since $u_gP_1u_g^{-1}$ is a self-adjoint indempotent, to show $P_2 = u_gP_1u_g^{-1}$ we need only show $\mathrm{ran}\,(u_gP_1u_g^{-1}) = \mathrm{ran}\,(gP_1g^{-1})$.

This is equivalent to showing $\mathrm{ran}\,(|g|P_1|g|^{-1}) = \mathrm{ran}\,(P_1)$, but $gP_1g^{-1} = P_2 = P_2^* = P_2P_2^*$ imply $\mathrm{ran}\,(|g|^2P_1(|g|^2)^{-1}) = \mathrm{ran}\,(P_1)$, and the identity then follows from (4.49).

To define the operators $g_r \in GL_\infty(H_Y)$ we modify an argument of [4] §15. To begin with, choose $\varepsilon \in (0,1)$ and suppose $\|P_1 - P_2\| \leq \varepsilon < 1$. Let

$$g_r = I + \frac{r}{\varepsilon}(P_2 - P_1)(P_1 - P_1^\perp) \ , \qquad 0 \leq r \leq \varepsilon \ .$$

Clearly, $P_2 g_\varepsilon = g_\varepsilon P_1$ and since $\|g_r - I\| < 1$ then $g_r$ is invertible. Moreover, since $P_1 - P_2$ is smoothing, so is $g_r - I$ and hence $g_r \in GL_\infty(H_Y)$. Since $D_{P_{i,0}}$ is invertible and invertibility is an open condition for continuous families of Fredholm operators, then by taking $\varepsilon$ smaller if necessary, $D_{P_{i,r}}$ will be invertible for $0 \leq r \leq \varepsilon$. Hence $g_r$ defines locally a path of the type required. Now $Gr_\infty^{(0)}(D)$ is path connected and hence for arbitrary $P, P' \in Gr_\infty^{(0)}(D)$ we can find a finite sequence $P_1 = P, \dots, P_m = P'$, in $Gr_\infty^{(0)}(D)$ with $\|P_i - P_{i+1}\| \leq \varepsilon_i$, $\varepsilon_i \in (0,1)$, for $i = 1, \dots, m-1$, and a finite sequence of paths $g_{r_i}$ in $GL_\infty(H_Y)$ with

$$P_i \leq g_{r_i} P_i g_{r_i}^{-1} \leq P_{i+1} \in \tilde{G}r_\infty(D) \ .$$

Finally, rescaling so that $0 \leq r_i \leq 1$ for each path, then $g_r = g_{r_{m-1}} \dots g_{r_1}$ is a path in $GL_\infty(H_Y)$ of the required form. This completes the proof.

**Proof of Lemma 4.12** Since we consider the simultaneous action of $U_\infty$ on $P_1, P_2$, we can 'gauge' transform the boundary variation to an order 0 variation of $\widehat{\Delta}$, and then appeal to Proposition 4.8. First, notice that the action

$$(g_r, P_i) \longmapsto g_r P_i g_r^{-1} = P_{i,r}$$

induces a dual action on the adjoint boundary condition

$$(\tilde{g}_r, P_i^\star) \longmapsto \tilde{g}_r P_i^\star \tilde{g}_r^{-1} = (g_r P_i g_r^{-1})^\star = P_{i,r}^\star \ ,$$

where $\tilde{g}_r = \sigma g_r \sigma^{-1}$. Moreover, with $\widehat{g}_r = \begin{pmatrix} g_r & 0 \\ 0 & \tilde{g}_r \end{pmatrix}$, we have

$$\widehat{g}_r \widehat{P}_i \widehat{g}_r^{-1} = P_{i,r} \oplus P_{i,r}^\star = \widehat{P_{i,r}} \ ,$$

and

$$\widehat{g}_r \widehat{\sigma} = \widehat{\sigma} \widehat{g}_r \ . \tag{4.50}$$

We can now transform the self-adjoint global boundary problem $\widehat{\Delta}_{\widehat{P}_{i,r}}$ to a unitary equivalent operator $\widehat{\Delta}(r)_{\widehat{P}}$ with constant domain by the method of [34, 28]. Let $f : [0,1] \to [0,1]$ be a non-decreasing function with $f(u) = 1$ for $u < 1/4$ and $f(u) = 0$ for $u > 3/4$. Then we extend $g_r$ and $\tilde{g}_r$ to unitary transformations

$$U_r = \begin{cases} g_{rf(u)} & \text{on} \quad \{u\} \times Y \subset U \\ Id & \text{on} \quad X \setminus U \end{cases} \quad , \quad \tilde{U}_r = \begin{cases} \tilde{g}_{rf(u)} & \text{on} \quad \{u\} \times Y = U \\ Id & \text{on} \quad X \setminus U \end{cases} \quad ,$$

on $L^2(X, E^1)$ and $L^2(X, E^2)$, respectively, and hence to a unitary transformation

$$\widehat{U}_r = \begin{cases} \widehat{g}_{rf(u)} & \text{on} \quad \{u\} \times Y \subset U \\ Id & \text{on} \quad X \setminus U \end{cases} = U_r \oplus \tilde{U}_r \ ,$$

on $L^2(X, E^1 \oplus E^2)$. Then

$$\widehat{\Delta}_{\widehat{P}_{i,r}} \quad \text{and} \quad \widehat{\Delta}(r)_{P_i} := (\widehat{U}_r^{-1} \widehat{\Delta} \widehat{U}_r)_{\widehat{P}_i}$$

are *unitarily equivalent*. Moreover, it is easy to check that

$$\widehat{\Delta}(r)_{P_i} = (\widehat{\Delta_r})_{\widehat{P}_i} \ , \tag{4.51}$$

where $\Delta_r = U_r^{-1} \Delta U_r = D_r^* D_r$ and $D_r$ is the Dirac-type operator $D_r = \tilde{U}_r^{-1} D U_r$.

Next, since $P(\widehat{\Delta}(r)_\mu) = \widehat{g}_r^{-1} P(\widehat{\Delta}_\mu) \widehat{g}_r$, from the multiplicativity of $\det_F$ we obtain

$$\det_F \left( \frac{S_\mu(\widehat{P}_{1,r})}{\widehat{\mathcal{E}} S_\mu(\widehat{P}_{2,r})} \right) = \det_F \left( \frac{S_{r,\mu}(\widehat{P}_1)}{\widehat{\mathcal{E}}_r S_{r,\mu}(\widehat{P}_2)} \right) \ ,$$

where $S_{r,\mu}(\widehat{P}_1) = \widehat{P}_1 \circ P(\widehat{\Delta}(r)_\mu)$ and $\mathcal{E}_r = \widehat{g}_r^{-1} \mathcal{E} \widehat{g}_r$. Hence from (4.19) and (4.51)

$$(d/dr) \log \det_\zeta(\Delta_{P_{1,r}}, \Delta_{P_{1,r}}) = (d/dr) \log \det_\zeta((\Delta_r)_{P_1}, (\Delta_r)_{P_2}) \ .$$

Finally, since (4.50) holds, then $\widehat{\Delta}(r)_{|U}$ has the form (3.30) with $\widehat{\mathcal{A}}_r = \widehat{g}_r^{-1} \widehat{\mathcal{A}} \widehat{g}_r^{-1}$, and since $\widehat{g}_r$ differs from the identity by a smoothing operator then $\sigma(\widehat{\mathcal{A}}_r)$ is independent of $r$. Hence we can apply Proposition 4.8 and the identity

$$\det_F(S(P_{i,r})^* S(P_{i,r})) = \det_F(S_r(P_i)^* S_r(P_i)) \ ,$$

which is a consequence of $P(D_r) = g_r^{-1} P(D) g_r$, to complete the proof.

**Proof of Lemma 4.4** We have $P(\widehat{\Delta}_\lambda) = \gamma \widehat{\mathcal{K}}_\lambda$, where (3.2)

$$\widehat{\mathcal{K}}_\lambda = \widehat{r} \widehat{\Delta}_{\lambda,d}^{-1} \widehat{\gamma}^* \widehat{\sigma} : H^{s-1/2}(Y, E^1_{|Y} \oplus E^2_{|Y}) \longrightarrow \mathrm{Ker}(\widehat{\Delta}_\lambda, s) \subset H^s(X, E^1 \oplus E^2) \ ,$$

and $\widehat{\Delta}_{\lambda,d}$ is an invertible operator over the double manifold $\tilde{X}$ with $(\widehat{\Delta}_{\lambda,d})_{|X} = \widehat{\Delta}_\lambda$, $\widehat{\gamma} = \gamma_1 \oplus \gamma_2$, $\widehat{r} = r_1 \oplus r_2$ with $\gamma_i : H^s(\tilde{X}, \tilde{E}^i) \to H^{s-1/2}(Y, E^i_{|Y})$, $r_i : H^s(\tilde{X}, \tilde{E}^i) \to H^s(X, E^i)$ the restriction operators, and $\widehat{\sigma}$ is defined in (4.7).

Let $D_d$ be the double operator of $D$, see for example [4]. Then $D_d$ is invertible on the closed double manifold $\tilde{X}$ with $(D_d)_{|X} = D$, and hence

$$\widehat{\Delta}_{\lambda,d} = \begin{pmatrix} -\lambda & D_d^* \\ D_d & -I \end{pmatrix}$$

is invertible on $\tilde{X}$ with $(\widehat{\Delta}_{\lambda,d})_{|X} = \widehat{\Delta}_\lambda$. We compute

$$\widehat{\Delta}_{\lambda,d}^{-1} = \begin{pmatrix} (\Delta_d - \lambda)^{-1} & D_d^*(\tilde{\Delta}_d - \lambda)^{-1} \\ D_d(\Delta_d - \lambda)^{-1} & \lambda(\tilde{\Delta}_d - \lambda)^{-1} \end{pmatrix} \ ,$$

where $\Delta_d = D_d^* D_d$, $\tilde{\Delta}_d = D_d D_d^*$, and hence that

$$\widehat{\mathcal{K}}_\lambda = \begin{pmatrix} r_1 D_d^*(\tilde{\Delta}_d - \lambda)^{-1} \gamma_1^* \sigma & r_1(\Delta_d - \lambda)^{-1} \gamma_0^* \sigma^\star \\ \lambda r_2(\tilde{\Delta}_d - \lambda)^{-1} \gamma_1^* \sigma & r_1 D_d(\Delta_d - \lambda)^{-1} \gamma_0^* \sigma^\star \end{pmatrix} \ ,$$

with $\sigma^\star := -\sigma^{-1}$. Setting $\lambda = 0$ we obtain

$$P(\widehat{\Delta}) = \gamma \begin{pmatrix} r_1 D_d^{-1}\gamma_1^*\sigma & r_1 D_d^{-1}(D_d^*)^{-1}\gamma_0^*\sigma^\star \\ 0 & r_1(D_d^*)^{-1}\gamma_0^*\sigma^\star \end{pmatrix} = \begin{pmatrix} \gamma r_1 D_d^{-1}\gamma_1^*\sigma & \gamma D_X^{-1}K_{D^*} \\ 0 & \gamma r_1(D_d^*)^{-1}\gamma_0^*\sigma^\star \end{pmatrix} ,$$

and since $D_X^{-1} = D_{P(D)}^{-1}$ we reach the conclusion.

## 5. An Application to Ordinary Differential Operators

In dimension one we can do better. Because no basepoint is needed to define the Grassmannian it is possible to apply the method of Theorem 2.5 to obtain formulas for the $\zeta$-determinant of individual boundary problems, rather than just relative formulas.

• **First-order operators.** We consider, as in §4, a first-order elliptic differential operator $D : C^\infty(X;E) \longrightarrow C^\infty(X;F)$, but where now $X = [0,\beta]$, $\beta > 0$, and $E, F$ are Hermitian bundles of rank $n$. Relative to trivializations of $E, F$ one has $D = A(x)d/dx + B(x)$, where $A(x), B(x)$ are complex $n \times n$ matrices and $A(x)$ is invertible. The restriction map to the boundary $Y = \{0\} \sqcup \{\beta\}$ is the map $\gamma : H^1(X;E) \longrightarrow E_0 \oplus E_\beta$, with $\gamma(\psi) = (\psi(0), \psi(\beta))$, and so global boundary conditions for $D$ are parameterized by the Grassmannian $Gr(E_0 \oplus E_\beta)$ of the $2n$-dimensional space of boundary 'fields'. For each $P \in Gr(E_0 \oplus E_\beta)$ we have a boundary problem $D_P : \mathrm{dom}(D_P) \longrightarrow L^2(X;E)$, where

$$\begin{aligned} \mathrm{dom}(D_P) &= \{\psi \in H^1(X;E) \mid P\gamma\psi = 0\} \\ &= \{\psi \in H^1(X;E) \mid M\psi(0) + N\psi(\beta) = 0\} \end{aligned}$$

and $[M\ N] \in \mathrm{Hom}(E_0 \oplus E_\beta, E_0)$ are Stiefel coordinates for $P$, see (3.62).

In dimension one, any element of $\mathrm{Ker}(D)$ has the form $K(x)v$ for some $v \in E_0$, where $K(x) \in \mathrm{Hom}(E_0, E_x)$ is the parallel transport operator uniquely solving $DK(x) = 0$ subject to $K(0) = I$. The isomorphism $\gamma : \mathrm{Ker}(D) \to H(D)$ is clear, while $H(D) = \mathrm{graph}(K : E_0 \to E_\beta) \subset E_0 \oplus E_\beta$, where $K := K(\beta)$.

Notice that $P(D) \in Gr_n(E_0 \oplus E_\beta)$ and hence the component $Gr_k(E_0 \oplus E_\beta)$ of the Grassmannian with $\mathrm{tr}\,(P) = k$ is the component with operator index $\mathrm{ind}\, D_P = \mathrm{ind}\, \mathcal{S}(P) = n-k$. The Poisson operator $\mathcal{K}_D : E_0 \oplus E_\beta \longrightarrow C^\infty(X,E)$ is the operator $\mathcal{K}_D(u)(x) = K(x)p_0P(D)u$, where $p_0$ is the projection map $E_0 \oplus E_\beta \to E_0$. It is easy to check that $D_{P(D)}^{-1}D = I - \mathcal{K}_D\gamma$ and hence that for invertible global boundary problems (3.19) holds:

$$D_{P_1}^{-1} - D_{P_2}^{-1} = -\mathcal{K}_D(P_1)P_1\gamma D_{P_2}^{-1} \ . \tag{5.1}$$

On the other hand, it is well known from elementary considerations that in Stiefel coordinates $D_P^{-1}$ has kernel

$$(5.2)\qquad k_P(x,y) = \begin{cases} -K(x)((M+NK)^{-1}NK)K(y)^{-1}A(y)^{-1} & x<y \\ \\ K(x)(I-(M+NK)^{-1}NK)K(y)^{-1}A(y)^{-1} & x>y\ , \end{cases}$$

Hence, if $P_1, P_2$ are represented by Stiefel coordinates $[M_1\ N_1], [M_2\ N_2]$, then the relative inverse $D_{P_1}^{-1} - D_{P_2}^{-1}$ has the smooth kernel

$$(5.3)\qquad -K(x)\left((M_1+N_1K)^{-1}N_1 - (M_2+N_2K)^{-1}N_2\right)K.K(y)^{-1}A(y)^{-1}\ ,$$

which can also be computed directly from (5.1) by using (3.60) with $\Phi(R) := \Pi(S(P))$.

We assume that $D_P$ is invertible with spectral cut $R_\theta$. For $\mathrm{Re}(s)>0$ we can then define $D_P^{-s} = \frac{i}{2\pi}\int_C \lambda_\theta^{-s}\,(D_P-\lambda)^{-1}\,d\lambda$ . Let $k_{P,\lambda}(x,y)$ be the kernel of $(D_P-\lambda)^{-1}$. From (5.2) one has $\lim_{\varepsilon\to 0}(k_{P,\lambda}(x,x+\varepsilon) - k_{P,\lambda}(x+\varepsilon,x) = -A(x)^{-1}$, and hence for $\mathrm{Re}(s)>1$ the kernel $p_s(x,y) = \frac{i}{2\pi}\int_C \lambda_\theta^{-s}\,k_{P,\lambda}(x,y)\,d\lambda$ of $D_P^{-s}$ is continuous, and $D_P^{-s}$ is trace class. Moreover, if $P(0)$ is the projection onto $E_0$, with Stiefel graph coordinates $[I\ 0]$, then from (5.2) we have $\mathrm{Tr}\,(D_{P(0)}^{-s}) = 0$. For $\mathrm{Re}(s)>1$,

$$(5.4)\qquad \zeta_\theta(D_P,s) = \zeta_\theta(D_P, D_{P(0)}, s)\ ,$$

is therefore a relative $\zeta$-function. Hence

$$\begin{aligned} \zeta_\theta(D_P,s) &= \frac{i}{2\pi}\int_C \lambda^{-s}\,\mathrm{Tr}\,((D_P-\lambda)^{-1} - (D_{P(0)}-\lambda)^{-1})\,d\lambda \\ &= -\frac{i}{2\pi}\int_C \lambda^{-s}\frac{\partial}{\partial\lambda}\log\det\left(\frac{S_\lambda(P)}{PS_\lambda(P(0))}\right)d\lambda \\ &= -\frac{i}{2\pi}\int_C \lambda^{-s}\frac{\partial}{\partial\lambda}\log\det(M+NK_\lambda)\,d\lambda\ , \end{aligned}$$

where $K_\lambda(x)$ is the solution operator for $D-\lambda$. The second equality is a restatement of (3.41), note $\det_F$ becomes the usual determinant here, and the third equality follows from (3.56) and Remark 3.18 (with $[M_1\ N_1] = [M\ N], [M_2\ N_2] = [I\ 0]$). Alternatively, with $\mathcal{M}_\lambda = M + NK_\lambda$, one can compute directly from (5.3)

$$\mathrm{Tr}\,((D_P-\lambda)^{-1} - (D_{P(0)}-\lambda)^{-1})$$

$$
\begin{aligned}
&= -\int_0^\beta \mathrm{tr}\,\left[K_\lambda(x)\mathcal{M}_\lambda^{-1}NK_\lambda K_\lambda(x)^{-1}A(x)^{-1}\right]\,dx \\
&= \int_0^\beta \mathrm{tr}\,\left[\frac{\partial}{\partial\lambda}(D-\lambda)K_\lambda(x)\mathcal{M}_\lambda^{-1}NK_\lambda K_\lambda(x)^{-1}A(x)^{-1}\right]\,dx \\
&= -\int_0^\beta \mathrm{tr}\,\left[(D-\lambda)\frac{\partial}{\partial\lambda}(K_\lambda(x))\mathcal{M}_\lambda^{-1}NK_\lambda K_\lambda(x)^{-1}A(x)^{-1}\right]\,dx \\
&= -\int_0^\beta \mathrm{tr}\,\left[\frac{d}{dx}\left(K_\lambda(x)^{-1}\frac{\partial}{\partial\lambda}(K_\lambda(x))\mathcal{M}_\lambda^{-1}NK_\lambda\right)\right]\,dx \\
&= -\frac{\partial}{\partial\lambda}\log\det\,\mathcal{M}_\lambda\ , \qquad (5.5)
\end{aligned}
$$

using $K_\lambda(x)^{-1}A(x)^{-1}(D-\lambda)K_\lambda(x) = d/dx$ and the $\lambda$ derivative of $(D-\lambda)K_\lambda(x) = 0$.

We now assume $D_P$ defines an elliptic boundary problem in the sense of [29]. Then there is an asymptotic expansion as $\lambda\to\infty$ in a sector $\Lambda_\theta$

$$
\mathrm{Tr}\,((D_P-\lambda)^{-1} - (D_{P(0)}-\lambda)^{-1}) \sim \sum_{j=1}^{\infty} b_j(-\lambda)^{-j}\ . \qquad (5.6)
$$

More precisely, $(D_{P_i}-\lambda)^{-2}$ is trace class and by ellipticity $\mathrm{Tr}\,((D_{P_i}-\lambda)^{-2}) \sim \sum_{j=1}^{\infty} a_j(-\lambda)^{-j-1}$ as $\lambda\to\infty$, and so applying (2.4) to the relative trace and observing that the trace class condition on the relative resolvent implies terms $(-\lambda)^{-\alpha_j}\log(-\lambda)$ with $\alpha_j \leq 0$ vanish, then (5.6) follows.

Thus $\zeta_\theta(D_P, D_{P(0)}, s)$ defines the meromorphic continuation of $\zeta_\theta(D_P, s)$ to $\mathbb{C}$ via the resolvent trace expansion (5.6), and this is regular at $s=0$. Hence we can define $\det_{\zeta,\theta}(D_P)$. In dimension one the following stronger variant of Theorem 3.17 (§4.4) holds:

**Theorem 5.1.** *Let $D_P$ be a first-order elliptic boundary problem over $[0,\beta]$ and let $[M\ N]$ be Stiefel coordinates for $P$. Then*

$$
\det_{\zeta,\theta}(D_P) = \det(M+NK)\,.\,e^{-\mathrm{LIM}^\theta_{\lambda\to\infty}\log\det(M+NK_\lambda)}\ . \qquad (5.7)
$$

*Invariantly, one has*

$$
\det_{\zeta,\theta}(D_P) = \det\left(\frac{S(P)}{PS(P(0))}\right)\,.\,e^{-\mathrm{LIM}^\theta_{\lambda\to\infty}\log\det\left(\frac{S_\lambda(P)}{PS_\lambda(P(0))}\right)}\ . \qquad (5.8)
$$

*Proof.* Immediate from (5.4), (5.5), (5.6) and Proposition 2.9 with $\Phi(\lambda) = \log\det\mathcal{M}_\lambda$, or from Theorem 2.5 with $\mathcal{S}_\lambda$ replaced by $\mathcal{M}_\lambda$. □

• **Operators of order $\geq 2$.** There is a straightforward generalization of these formulas to differential operators $D : C^\infty(X,E)\to C^\infty(X,F)$ of order $r\geq 2$. With respect to trivializations of $E, F$

$$
D = \sum_{k=0}^{r} B_k(x)\frac{d^k}{dx^k} : C^\infty(X;\mathbb{C}^n)\to C^\infty(X;\mathbb{C}^n)\ ,
$$

with complex matrix coefficients and $\det B_r(x) \neq 0$. The restriction map is

$$\gamma_{r-1} : H^r(X;E) \longrightarrow \mathbb{C}^{rn} \oplus \mathbb{C}^{rn} \ , \quad \gamma_{r-1}(\psi) = (\widehat{\psi}(0), \widehat{\psi}(\beta)) \ , \tag{5.9}$$

where $\widehat{\psi}(x) = (\psi(x), \dots, \psi^{(r-1)}(x))$. The form of $\gamma_{r-1}$ means that one can study boundary problems for $D$ through the first-order system on $C^\infty(X;\mathbb{C}^{rn})$

$$\widehat{D} = \frac{d}{dx} - \begin{bmatrix} 0 & I & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I \\ -B_r(x)^{-1}B_0(x) & -B_r(x)^{-1}B_1(x) & \dots & -B_r(x)^{-1}B_{r-1}(x) \end{bmatrix} . \tag{5.10}$$

This is well known [8, 15]. $\widehat{D}$ extends to a continuous map $H^1(X;\mathbb{C}^{rn}) \to L^2(X;\mathbb{C}^{rn})$, and with respect to the inclusion $\hat{\imath} : H^r(X;\mathbb{C}^n) \longrightarrow H^1(X;\mathbb{C}^{rn})$, $\psi \mapsto \widehat{\psi}$, we have $\gamma_{r-1} = \gamma \circ \hat{\imath}$. More precisely, there is an isomorphism $\mathrm{Ker}(D) \cong \mathrm{Ker}(\widehat{D})$, for from

$$\widehat{D}(\psi, \dots, \psi^{(r-1)}) = (0, \dots, 0, B_r^{-1}D\psi) \ , \tag{5.11}$$

a basis $\{\psi_1, \dots, \psi_k\}$ for $\mathrm{Ker}(D)$ defines a basis $\{\widehat{\psi}_1, \dots, \widehat{\psi}_k\}$ for $\mathrm{Ker}(\widehat{D})$. A solution of $\widehat{D}$ is characterized by its parallel transport operator $\widehat{K}(x)$, as before, the columns of which are a preferred basis for $\mathrm{Ker}(\widehat{D})$. One has

$$\mathrm{ran}\,(P(\widehat{D})) = \mathrm{graph}(\widehat{K} : \mathbb{C}^{rn} \to \mathbb{C}^{rn})$$

and $\widehat{D}$ has Poisson operator $\widehat{\mathcal{K}} : \mathbb{C}^{rn} \oplus \mathbb{C}^{rn} \to C^\infty(X;\mathbb{C}^{rn})$ defined by

$$\widehat{\mathcal{K}}(v)(x) = \widehat{K}(x) p_0 P(\widehat{\Delta}) v \ .$$

A global boundary condition for $D$ is defined by a global boundary condition $P \in Gr(\mathbb{C}^{rn} \oplus \mathbb{C}^{rn})$ for $\widehat{D}$. That is

$$\begin{aligned} \mathrm{dom}(D_P) &= \{\psi \in H^r(X;E) \mid P\gamma_{r-1}\psi = 0\} \\ &= \{\psi \in H^r(X;E) \mid \widehat{M}\widehat{\psi}(0) + \widehat{N}\widehat{\psi}(\beta) = 0\} \ , \end{aligned}$$

where $[\widehat{M}, \widehat{N}]$ are Stiefel coordinates for $P$.

The boundary problem $D_P$ is modeled by the finite-rank operator on boundary data $\widehat{S}(P) := P \circ P(\widehat{D}) : K(\widehat{D}) \to \mathrm{ran}\,(P)$. From (5.11) we have that $D_P$ is invertible if and only if $\widehat{D}_P$ is invertible, and in that case

$$D_P^{-1} = [\widehat{D}_P^{-1}]_{(1,r)} B_r^{-1} : L^2(X;\mathbb{C}^n) \to \mathrm{dom}\,(D_P) \ .$$

Here $[\widehat{D}_P^{-1}]_{(1,r)}$ means the integral operator $\int_0^\beta [\widehat{k}(x,y)]_{(1,r)} \psi(y)\, dy$ where $\widehat{k}(x,y)$ is the kernel of $\widehat{D}_P^{-1}$, and, as in [15], $[T]_{(1,r)}$ is the $n \times n$ matrix in the $(1,r)^{\mathrm{th}}$ position in an $r \times r$ block matrix $T \in \mathrm{End}(\mathbb{C}^{rn})$. For $D_{P_1}, D_{P_2}$ invertible, this leads to the formula

$$D_{P_1}^{-1} = D_{P_2}^{-1} - [\widehat{\mathcal{K}}\widehat{S}(P_1)^{-1} P_1 \gamma \widehat{D}_{P_2}^{-1}]_{(1,r)} B_r^{-1} \ . \tag{5.12}$$

In Stiefel coordinates $D_P^{-1}$ has kernel

$$k_P(x,y) = \begin{cases} -[\widehat{K}(x)(\widehat{\mathcal{M}}^{-1}\widehat{N}\widehat{K})\widehat{K}(y)^{-1}]_{(1,r)}B_r(y)^{-1} & x < y \\ \\ [\widehat{K}(x)(I - \widehat{\mathcal{M}}^{-1}\widehat{N}\widehat{K})\widehat{K}(y)^{-1}]_{(1,r)}B_r(y)^{-1} & x > y, \end{cases} \tag{5.13}$$

where $\widehat{\mathcal{M}} = \widehat{M} + \widehat{N}\widehat{K}$.

Since we are in dimension one, the resolvent of a differential operator of order $r \geq 2$ is trace class (as is evident from (5.13)). Let $R_\theta$ be a spectral cut for $D_P$ and let $P$ now be a local elliptic boundary condition for $D$ [29]. Then Seeley proved [32] that as $\lambda \to \infty$ in $\Lambda_\theta$ the resolvent trace has an asymptotic expansion

$$\mathrm{Tr}\,((D_P - \lambda)^{-1}) \sim \sum_{j=-1}^{\infty} b_j(-\lambda)^{-j/r-1}\ . \tag{5.14}$$

On the other hand, $(D-\lambda)_P^{-1} = [\widehat{D}_{\lambda,P}^{-1}]_{(1,r)}B_r^{-1}$ with $\widehat{D}_\lambda = (\widehat{D-\lambda})$, with $\widehat{K}_\lambda(x)$ the parallel transport operator for $\widehat{D}_\lambda$ and $\widehat{\mathcal{M}}_\lambda = \widehat{M} + \widehat{N}\widehat{K}_\lambda$

$$\begin{aligned} \mathrm{Tr}\,((D_P - \lambda)^{-1}) &= \int_0^\beta \mathrm{tr}\,\left[\frac{\partial}{\partial\lambda}(\widehat{D}_\lambda)\widehat{K}_\lambda(x)\widehat{\mathcal{M}}_\lambda^{-1}\widehat{N}\widehat{K}_\lambda\widehat{K}_\lambda(x)^{-1}\right] dx \\ &= -\frac{\partial}{\partial\lambda}\log\det\,\widehat{\mathcal{M}}_\lambda\ , \end{aligned}$$

This follows by the same argument as before, using the device

$$\mathrm{tr}\,([T(x)]_{(1,r)}) = \mathrm{tr}\,(JT(x)) = -\mathrm{tr}\,(\frac{\partial}{\partial\lambda}(\widehat{D}_\lambda)T(x))\ ,$$

where $J$ is the $n\times n$ block matrix with the identity in the $(r,1)^{\text{th}}$ position and zeroes elsewhere.

By Proposition 2.9 we therefore obtain the extension of Theorem 5.1 to higher order operators:

**Theorem 5.2.** *Let $D_P$ be a local elliptic boundary problem of order $r \geq 2$ over $[0,\beta]$ and let $[\widehat{M}\ \widehat{N}]$ be Stiefel coordinates for $P \in Gr(\mathbb{C}^{rn} \oplus \mathbb{C}^{rn})$. Then*

$$\det_{\zeta,\theta}(D_P) = \det(\widehat{M} + \widehat{N}\widehat{K})\,.\,e^{-\mathrm{LIM}^\theta_{\lambda\to\infty}\log\det\left(\widehat{M}+\widehat{N}\widehat{K}_\lambda\right)}\ . \tag{5.15}$$

*Invariantly, one has*

$$\det_{\zeta,\theta}(D_P) = \det\left(\frac{\widehat{S}(P)}{P\widehat{S}(P(0))}\right)\,.\,e^{-\mathrm{LIM}^\theta_{\lambda\to\infty}\log\det\left(\frac{\widehat{S}_\lambda(P)}{P\widehat{S}_\lambda(P(0))}\right)}\ ,$$

*where $P(0)$ is the projection to the first factor in $\mathbb{C}^{rn}\oplus\mathbb{C}^{rn}$.*

The formulas (5.7) and (5.15) were first proved in [15].

## References


[1] Atiyah, M.F., Patodi, V.K. and Singer, I.M.: 1975, 'Spectral asymmetry and Riemannian geometry. I', *Math. Proc. Cambridge Phil. Soc.* **77**, 43–69.

[2] Berline, N., E. Getzler, and M. Vergne: *Heat Kernels and Dirac Operators.* Grundlehren der Mathematischen Wissenschaften **298**, Springer-Verlag, Berlin, 1992.

[3] Bismut, J-M and Freed, D.S.: 1986, D., The analysis of elliptic families I, *Comm. Math. Phys.* **106**, 159–176.

[4] Booß–Bavnbek, B., and Wojciechowski, K.P.: 1993, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston.

[5] Brünning, J. and Lesch, M.: 1999, On the $\eta$-invariant of certain nonlocal boundary value problems, *Duke Math. J.* **96**, 425–468.

[6] Dai, X., and Zhang, X.: 1997, 'The splitting of the family index', *Comm. Math. Phys.* **35**, 5155–5195.

[7] Douglas, R.G., and Wojciechowski, K.P.: 1991, 'Adiabatic limits of the $\eta$-invariants. The odd-dimensional Atiyah-Patodi-Singer problem', *Comm. Math. Phys.* **142**, 139–168.

[8] Forman R.: 1987, 'Functional determinants and geometry', *Invent. Math.***88**, 447–493.

[9] Grubb, G.: 1996, *Functional Calculus of Pseudodifferential Boundary Problems*, Progr. Math. **65**, 2nd ed., Birkhauser, Boston, Mass.

[10] Grubb, G.: 1999, 'Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems', *Ark. Mat.* **37**, 45–86.

[11] Grubb, G.: 2001, 'Poles of zeta and eta functions for perturbations of the Atiyah-Patodi-Singer Problem', *Commun. Math. Phys.* **215**, 583-589

[12] Grubb, G., and Seeley. R.: 1996, 'Zeta and eta functions for Atiyah-Patodi-Singer operators', *J. Geom. Anal.* **6**, 31–77.

[13] Kontsevitch, M. and Vishik, S.: 1993, *'Geometry of determinants of elliptic operators'.* In: Progr. Math. **131**, Vol. I, 173–197, Birkhauser, Boston, Mass.

[14] Kirk, P. and Lesch, M.: 2000, 'The $\eta$–invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary', *preprint.* math.DG/0012123.

[15] Lesch, M., Tolksdorf, J.: 1998, 'On the determinant of one-dimensional elliptic boundary value problems', *Comm. Math. Phys.* **193**, 643–660.

[16] Lesch, M. and Wojciechowski, K.P.: 1996, 'On the $\eta$–invariant of generalized Atiyah–Patodi–Singer problems', *Illinois J. Math.* **40**, 30–46.

[17] Melrose, R.B.: 1995, 'The eta invariant of pseudodifferential operators', *Math. Res. Letters* **2**, 541–561.

[18] Melrose, R.B., Piazza, P.: 1997, 'Families of Dirac operators, boundaries and the $b$-calculus', *J. Diff. Geom.* **46**, 99–167.

[19] Mickelsson, J. and Scott, S.G.: 2001 'Functorial QFT, gauge anomalies and the Dirac determinant bundle', *Comm. Math. Phys.* **219**, 567–605.

[20] Müller, W.: 1998, 'Eta invariants and manifolds with boundary', *J. Diff. Geom.* **40**, 311–377.

[21] Müller, W.: 1998, 'Relative zeta functions, relative determinants and scattering theory', *Comm. Math. Phys.* **192**, 309–347.

[22] Pressley, A. and Segal, G.B.: *Loop Groups.* Oxford: Clarendon Press, 1986.

[23] Quillen, D.G: 1988, 'Superconnection character forms and the Cayley transform', *Topology* **23**, 211–238.

[24] Scott, S.G.: 1995, 'Determinants of Dirac boundary value problems over odd–dimensional manifolds', *Comm. Math. Phys.* **173**, 43–76.

[25] Scott, S.G.: 2000, 'Splitting the curvature of the determinant line bundle', *Proc. Am. Math. Soc.* **128**, 2763–2775.

[26] Scott, S.G.: 2001, 'Relative zeta determinants and the geometry of the determinant line bundle', *Electron. Res. Announc. Amer. Math. Soc.* **7**, 8–16.

[27] Scott, S.G.: 'Eta forms and the Chern character'. Preprint.

[28] Scott, S.G., and Wojciechowski, K.P.: 2000, 'The $\zeta$–Determinant and Quillen's determinant for a Dirac operator on a manifold with boundary', *Geom. Funct. Anal.* **10**, 1202–1236, 2000.

[29] Seeley, R. T.: 1966, 'Singular integrals and boundary value problems', *Amer. J. Math.* **88**, 781–809.

[30] Seeley, R. T.: 1967, 'Complex powers of an elliptic operator', AMS *Proc. Symp. Pure Math. X.* AMS Providence, 288–307.

[31] Seeley, R. T.: 1969, *'Topics in pseudodifferential operators'.* In: *CIME Conference on Pseudo-Differential operators (Stresa 1968),* pp. 167–305. Cremonese 1969.

[32] Seeley, R. T.: 1969, 'Analytic extension of the trace associated with elliptic boundary problems', *Amer. J. Math.* **91**, 963–983.

[33] Segal, G.B.: c1990, Oxford Lectures.

[34] Wojciechowski, K.P.: 1999, 'The $\zeta$-determinant and the additivity of the $\eta$-invariant on the smooth, self-adjoint Grassmannian', *Comm. Math. Phys.* **201**, 423–444.



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